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Matricial Method Used for Reaction Forces Computation in the Case of Continous Beams Supported by Uneven Bearings

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Abstract

The actual methods used in the analysis of the static indeterminate structures as continuous beams supported by n stiff bearings, are mostly based on writing the continuity conditions of the angular deformations in the beam's supports for all the pairs concurrent beam spans which meet in supports 2, 3, 4, .. n-1. In this way one obtains CLAPEYRON equations or the Three Moment Equation (M_{n-1}, M_n, M_{n+1}). Solving the n-2 equations (the bending efforts from the first and last support being known) one obtains the unknown bending efforts. The reaction forces are obtained by superposing the effects of the exterior loads and interior ones for each beam span. The matricial method proposed by this work uses the MATCAD step function $\Phi(x-a)$, which allows the unitary form analytical expression of the functions: shear efforts T(x), bending efforts M(x), angular deformations $\varphi(x)$ and displacements w(x) depending on the exterior loads applied on the structure (known) and on the reaction forces from the supports (unknown).

Key words: *matricial method , step function, continuous beams*

Continous Beam Supported by Four Uneven Stiff Bearings

One considers the continuous beam *OA* supported by four stiff bearings, having different levels with respect to the beam's axis: $w_1=0,\ w_2\neq 0,\ w_3\neq 0,\ w_4=0$. The beam has a constant bending stiffness *EI,* along its whole length. The beam's exterior loading is known: *N, P,* q_0 and q_1. The support reactions V_1, V_2, V_3 and V_4 are unknown (fig.1).

In order to compute the values of the reaction forces V_1, V_2, V_3 and V_4 one uses two equilibrium equations from Mechanics:

$$\begin{aligned}&\sum F_{zs}\downarrow = V_1+V_2+V_3+V_4\\&\sum \overset{\leftarrow}{M}_{4s} = V_1(r_4-r_1)+V_2(r_4-r_2)+V_3(r_4-r_3)\end{aligned}\ , \tag{1}$$

where: $\sum F_{zs}\downarrow$ is the sum of exterior forces on *Oz* axis*;*

$\sum \overset{\leftarrow}{M}_{4s}$ is the sum of moments and exterior force couples with respect to the *Oy* axis which passes through point *4*

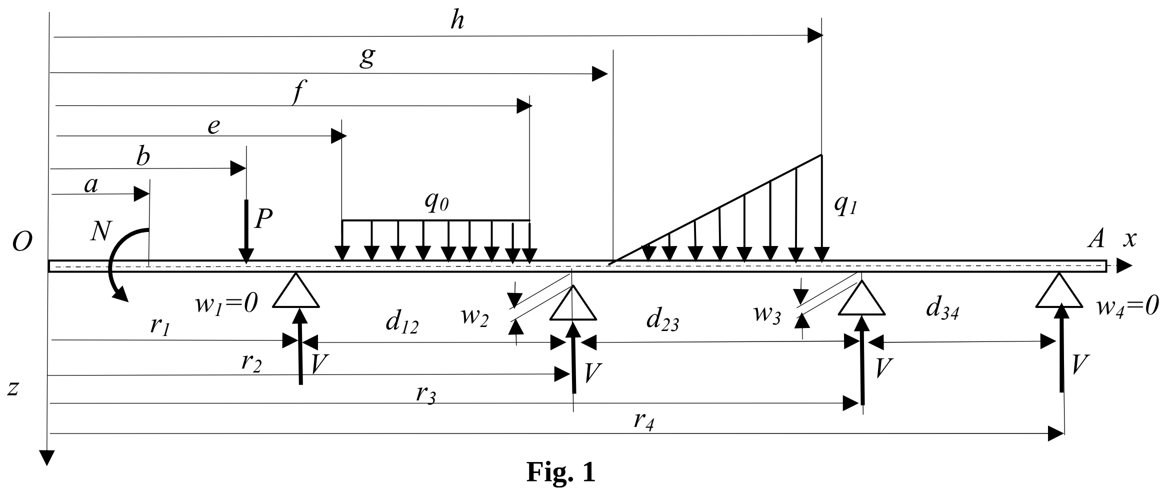

Fig. 1

The third and the fourth equations will result by replacing the origin parameters (w_0 and φ_0) in the displacement equations from the supports:

$$
\begin{aligned}
&w_0+\varphi_0\cdot r_1+\frac{1}{EI}W(r_1)=w_1\\
&w_0+\varphi_0\cdot r_2+\frac{1}{EI}W(r_2)=w_2\\
&w_0+\varphi_0\cdot r_3+\frac{1}{EI}W(r_3)=w_3\\
&w_0+\varphi_0\cdot r_4+\frac{1}{EI}W(r_4)=w_4
\end{aligned}
\tag{2}
$$

where $W(x)$ is the second integral of the bending efforts with changed sign for the four types of known efforts as well as for the unknown reaction forces V_1, V_2, V_3 and V_4(fig.1):

$$
\begin{aligned}
W(x)=\;&N\cdot\Phi(x-a)\cdot\frac{(x-a)^2}{2}+P\cdot\Phi(x-b)\cdot\frac{(x-b)^3}{6}+q_0\cdot\Phi(x-e)\cdot\frac{(x-e)^4}{24}-\\
&-q_0\cdot\Phi(x-f)\cdot\frac{(x-f)^4}{24}+q_1\cdot\Phi(x-g)\cdot\frac{(x-g)^5}{120(h-g)}-q_1\cdot\Phi(x-h)\cdot\frac{(x-h)^4}{24}-q_1\cdot\Phi(x-h)\cdot\frac{(x-h)^5}{120(h-g)}-\\
&-V_1\cdot\Phi(x-r_1)\cdot\frac{(x-r_1)^3}{6}-V_2\cdot\Phi(x-r_2)\cdot\frac{(x-r_2)^3}{6}-V_3\cdot\Phi(x-r_3)\cdot\frac{(x-r_3)^3}{6}.
\end{aligned}
\tag{3}
$$

Replacing w_0 and φ_0 in the first and last couple of equations (2) one obtains:

$$
\begin{aligned}
&\frac{W(r_2)-W(r_1)}{d_{12}}-EI\frac{w_2-w_1}{d_{12}}=\frac{W(r_3)-W(r_2)}{d_{23}}-EI\frac{w_3-w_2}{d_{23}}\\
&\frac{W(r_3)-W(r_2)}{d_{23}}-EI\frac{w_3-w_2}{d_{23}}=\frac{W(r_4)-W(r_3)}{d_{34}}-EI\frac{w_4-w_3}{d_{34}}
\end{aligned}
\tag{4}
$$

where: $d_{12}=r_2-r_1$, distance between supports 1 and 2.

$d_{23}=r_3-r_2$, distance between supports 2 and 3

$d_{34}=r_4-r_3$, distance between supports 3 and 4.

If one denotes with *Ws(x)* the second integral of the bending efforts with changed sign written only for known exterior loads:

$$
\begin{aligned}
&Ws(x) = N\cdot\Phi(x-a)\cdot\frac{(x-a)^2}{2} + P\cdot\Phi(x-b)\cdot\frac{(x-b)^3}{6} + q_0\cdot\Phi(x-e)\cdot\frac{(x-e)^4}{24} -\\
&-q_0\cdot\Phi(x-f)\cdot\frac{(x-f)^4}{24} + q_1\cdot\Phi(x-g)\cdot\frac{(x-g)^5}{120(h-g)} - q_1\cdot\Phi(x-h)\cdot\frac{(x-h)^4}{24} - q_1\cdot\Phi(x-h)\cdot\frac{(x-h)^5}{120(h-g)}.\\
&\text{and}\quad k_1(x)=\Phi(x-r_1)\cdot\frac{(x-r_1)^3}{6};\quad k_2(x)=\Phi(x-r_2)\cdot\frac{(x-r_2)^3}{6};\ k_3(x)=\Phi(x-r_3)\cdot\frac{(x-r_3)^3}{6}\\
&k_4(x)=\Phi(x-r_4)\cdot\frac{(x-r_4)^3}{6};\\
&k_1(r_1)=k_{11};\ k_1(r_2)=k_{12};\ k_2(r_1)=k_{21},\ \text{and so on}\\
&Ws(r_1)=W_{s1};\ Ws(r_2)=W_{s2};\ Ws(r_3)=W_{s3},\ \text{and so on}
\end{aligned}
\tag{5}
$$

Isolating the unknowns V_1, V_2, V_3 and V_4 from equations (4) one obtains:

$$
\begin{aligned}
&V_1\cdot\left(\frac{k_{11}-k_{12}}{d_{12}}-\frac{k_{12}-k_{13}}{d_{23}}\right)+V_2\cdot\left(\frac{k_{21}-k_{22}}{d_{12}}-\frac{k_{21}-k_{23}}{d_{23}}\right)+V_3\cdot\left(\frac{k_{31}-k_{32}}{d_{12}}-\frac{k_{32}-k_{33}}{d_{23}}\right)=\\
&\qquad=\frac{W_{s1}-W_{s2}}{d_{12}}-\frac{W_{s2}-W_{s3}}{d_{23}}-EI\frac{w_1-w_2}{d_{12}}+EI\frac{w_2-w_3}{d_{23}}\\
&V_1\cdot\left(\frac{k_{12}-k_{13}}{d_{23}}-\frac{k_{12}-k_{13}}{d_{34}}\right)+V_2\cdot\left(\frac{k_{22}-k_{23}}{d_{23}}-\frac{k_{23}-k_{24}}{d_{34}}\right)+V_3\cdot\left(\frac{k_{32}-k_{33}}{d_{23}}-\frac{k_{33}-k_{34}}{d_{34}}\right)+\\
&+V_4\cdot\left(\frac{k_{42}-k_{43}}{d_{23}}-\frac{k_{42}-k_{43}}{d_{34}}\right)=\frac{W_{s2}-W_{s3}}{d_{23}}-\frac{W_{s3}-W_{s4}}{d_{34}}-EI\frac{w_2-w_3}{d_{23}}+EI\frac{w_3-w_4}{d_{34}}
\end{aligned}
\tag{6}
$$

The equations (1) and (6) allow the computation of V_1, V_2, V_3 and V_4. They are written in matricial form as it follows:

$$
\begin{bmatrix}
1 & 1 & 1 & 1\\
d_{12}+d_{23}+d_{34} & d_{23}+d_{34} & d_{34} & 0\\
\frac{k_{11}-k_{12}}{d_{12}}-\frac{k_{12}-k_{13}}{d_{23}} & \frac{k_{21}-k_{22}}{d_{12}}-\frac{k_{22}-k_{23}}{d_{23}} & \frac{k_{31}-k_{32}}{d_{12}}-\frac{k_{32}-k_{33}}{d_{23}} & 0\\
\frac{k_{12}-k_{13}}{d_{23}}-\frac{k_{13}-k_{14}}{d_{34}} & \frac{k_{22}-k_{23}}{d_{23}}-\frac{k_{23}-k_{24}}{d_{34}} & \frac{k_{32}-k_{33}}{d_{23}}-\frac{k_{33}-k_{34}}{d_{34}} & \frac{k_{42}-k_{43}}{d_{23}}-\frac{k_{43}-k_{44}}{d_{34}}
\end{bmatrix}
\begin{Bmatrix} V_1\\ V_2\\ V_3\\ V_4\end{Bmatrix}=
$$

$$
=\begin{Bmatrix}
\sum F_{zs}\downarrow\\
\sum \overleftarrow{M}_{4s}\\
\frac{W_{s1}-W_{s2}}{d_{12}}-\frac{W_{s2}-W_{s3}}{d_{23}}-EI\frac{w_1-w_2}{d_{12}}+EI\frac{w_2-w_3}{d_{23}}\\
\frac{W_{s2}-W_{s3}}{d_{23}}-\frac{W_{s3}-W_{s4}}{d_{34}}-EI\frac{w_2-w_3}{d_{23}}+EI\frac{w_3-w_4}{d_{34}}
\end{Bmatrix}
\tag{7}
$$

Numerical application

The following continuous beam is supported on four stiff bearings with different levels with respect to the beam's axis: $w_1=0$, $w_2=0.02\ m$, $w_3=0.05m$ *and* $w4=0$. The beam has a constant bending stiffness $EI=1000\ kN\ m^2$ along its entire length and is loaded as shown in figure 5. It is asked to conduct the following tasks:

- Determination of reaction forces V_1, V_2, V_3, V_4
- Plot of the bending efforts diagram
- Plot of the deformed shape of the beam.

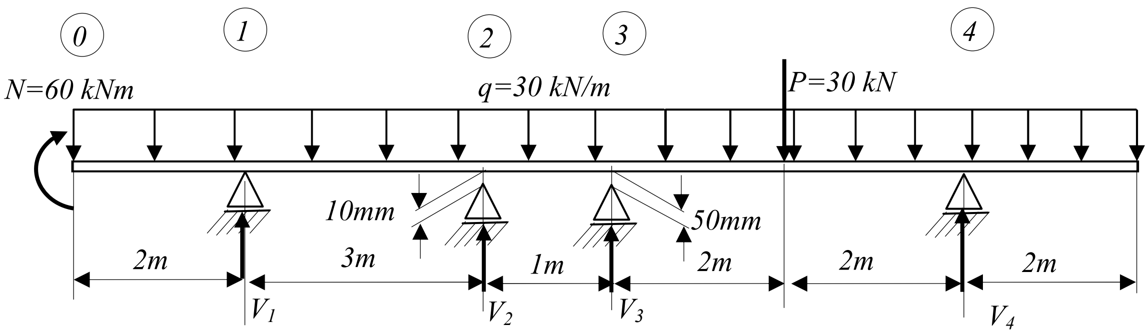

Fig 2

In order to determine the reaction forces V_1, V_2 V_3, V_4 one introduces the input data and the matricial equation *(7)* in MATCAD and the following results occur:

INPUT DATA $r1 := 2$ $\quad r2 := 5$ $\quad r3 := 6$ $\quad r4 := 10$ $\quad EI := 1000$

$d1 := 8$ $\quad P1 := 30$

$e1 := 0$ $\quad f1 := 12$ $\quad Q1 := 30$

$g1 := 0$ $\quad N1 := -60$

$w1 := 0$ $\quad w2 := 0.02$ $\quad w3 := 0.05$ $\quad w4 := 0$

REACTION FORCES COMPUTATION

$d12 := r2 - r1$ $\qquad d23 := r3 - r2$ $\qquad d34 := r4 - r3$

$$Ws(x) := \frac{N1}{2}\cdot\Phi(x-g1)\cdot(x-g1)^2 + \frac{P1}{6}\cdot\Phi(x-d1)\cdot(x-d1)^3 + \frac{Q1}{24}\cdot\Phi(x-e1)\cdot(x-e1)^4 - \frac{Q1}{24}\cdot\Phi(x-f1)\cdot(x-f1)^4$$

$$k1(x) := \Phi(x-r1)\cdot\frac{(x-r1)^3}{6} \qquad k2(x) := \Phi(x-r2)\cdot\frac{(x-r2)^3}{6} \qquad k3(x) := \Phi(x-r3)\cdot\frac{(x-r3)^3}{6}$$

$$k4(x) := \Phi(x-r4)\cdot\frac{(x-r4)^3}{6}$$

$$M := \begin{pmatrix} 1 & 1 & 1 & 1 \\ d12 + d23 + d34 & d23 + d34 & d34 & 0 \\ \frac{k1(r1) - k1(r2)}{d12} - \frac{k1(r2) - k1(r3)}{d23} & \frac{k2(r1) - k2(r2)}{d12} - \frac{k2(r2) - k2(r3)}{d23} & \frac{k3(r1) - k3(r2)}{d12} - \frac{k3(r2) - k3(r3)}{d23} & 0 \\ \frac{k1(r2) - k1(r3)}{d23} - \frac{k1(r3) - k1(r4)}{d34} & \frac{k2(r2) - k2(r3)}{d23} - \frac{k2(r3) - k2(r4)}{d34} & \frac{k3(r2) - k3(r3)}{d23} - \frac{k3(r3) - k3(r4)}{d34} & \frac{k4(r2) - k4(r3)}{d23} - \frac{k4(r3) - k4(r4)}{d34} \end{pmatrix}$$

$$\text{lsolve}(M, v) = \begin{pmatrix} 90.960 \\ 103.122 \\ 49.177 \\ 146.741 \end{pmatrix}$$

$$v := \begin{bmatrix} P1 + Q1\cdot(f1 - e1) \\ N1 + P1\cdot(r4 - d1) + Q1\cdot(f1 - e1)\cdot(r4 - 0.5e1 - 0.5f1) \\ \frac{Ws(r1) - Ws(r2)}{d12} - \frac{Ws(r2) - Ws(r3)}{d23} - \frac{EI\cdot(w1 - w2)}{d12} + \frac{EI\cdot(w2 - w3)}{d23} \\ \frac{Ws(r2) - Ws(r3)}{d23} - \frac{Ws(r3) - Ws(r4)}{d34} - \frac{EI\cdot(w2 - w3)}{d23} + \frac{EI\cdot(w3 - w4)}{d34} \end{bmatrix}$$

The bending efforts diagram and the deformed shape (amplified 200 times for better visualization of the displacements) corresponding to the continuous beam supported by four bearings and loaded as depicted in figure 2 are plotted in figure 3.

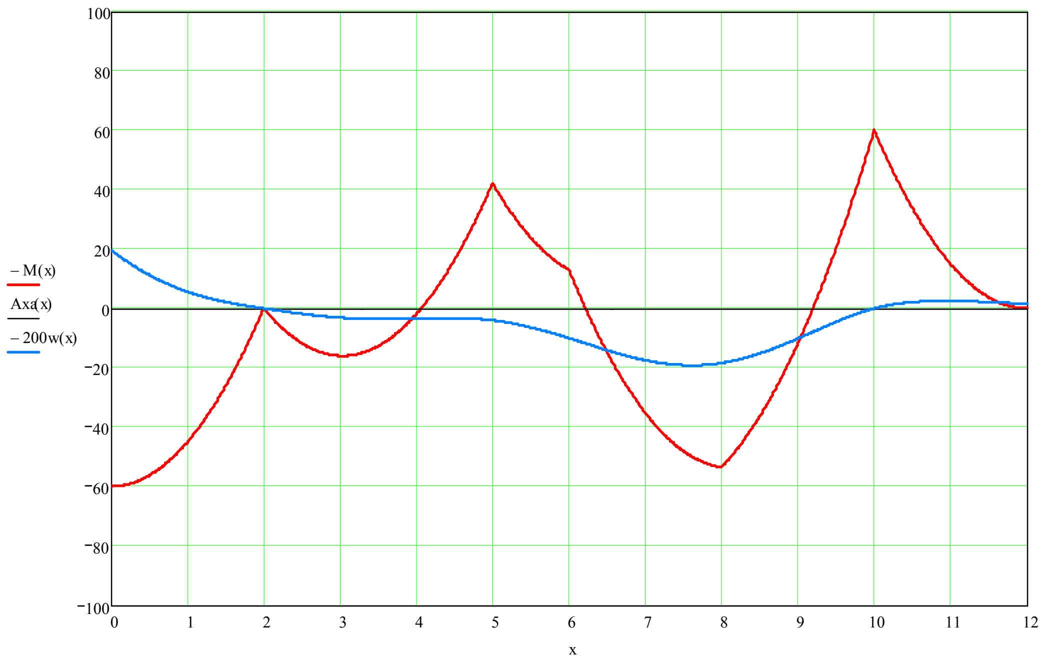

Fig. 3.

Conclusions

- This method is appropriate for strength check of continuous beams using modern computation solutions considering the fact that it uses matricial calculus. Hence if one introduces in MATCAD the input data of the parameters and the mathematical relations, the numerical values of the reactions, the diagram of bending efforts and the displaced shape of the beam are obtained instantaneous.
- Using the step function this method can be easily generalized for a larger number of supports due to the symmetry of the matrices M and v presented above.
- This method can be particularized for the special case of continuous beams supported by even bearings by simply imposing the conditions: $w_1=0$, $w_2=0$, $w_3=0$ and $w_4=0$

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Metodă matriceală pentru calculul reacțiunilor pentru grinzile continue situate pe reazeme denivelate

Rezumat

Metodele grafice folosite în prezent pentru rezolvarea sistemelor static nedeterminate de tipul grinzilor continue situate pe n reazeme punctuale rigide se bazează pe scrierea condițiilor de continuitate a deformațiilor unghiulare în reazemele grinzii, pentru toate perechile de tronsoane concurente în reazemele 2, 3, 4, .. n-1, obținându-se astfel ecuațiile lui CLAPEYRON sau ecuația celor trei momente M_{n-1}, M_n *şi* M_{n+1}. *Prin rezolvarea celor n-2 ecuații (eforturile încovoietoare din primul şi ultimul reazem sunt cunoscute) se obțin eforturile încovoietoare necunoscute. Reacțiunile se determină prin suprapunerea efectelor sarcinilor exterioare şi a eforturilor interioare pentru fiecare tronson al grinzii continue. Metoda matriceală propusă în această lucrare pentru calculul reacțiunilor foloseşte funcția MATCAD de tip treaptă* $\Phi(x-a)$ *care permite exprimarea analitică sub o formă unitară a funcțiilor: eforturi tăietoare* $T(x)$, *eforturi încovoietoare* $M(x)$, *deformații unghiulare* $\varphi(x)$ *şi săgeți* $w(x)$ *în funcție de sarcinile exterioare aplicate (cunoscute) şi a reacțiunilor din reazeme (necunoscute).*