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# Some Aspects Regarding the Deflection of the Beams with Variable Cross Sectional Area 

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#### Abstract

In the paper is presented a way of calculation of the maximum deflection for a beam with a variable cross sectional area. When such a beam is externally loaded with a concentrated force the maximum deflection is ussually reached in another section. In this paper is presented the mrethod of finding the section where the deflection has the maximum value in the same point where the external force acts. This section is dangerous especially for the transversal impact and is recommended to be avoided. The results obtained are analysed in a calculus example.


Key words: deflection, elastic curve.

## General Equations

In order to develop some concrete equations a beam with a variable cross sectional area is considered (fig. 1).


Fig. 1. Beam with a variable cross sectional area

The beam has a linear depth $b_{x}$, the length $l$ and the thickness $h$. In a current section specified by a current abscissa $x$ acts the external concentrated force $F$. The beam is simply supported and its depth presents a liniar variation along the length. If the depth in the origin is $\mathrm{b}_{0}$ the current depth of the beam can be expressed from the geometry of the beam as : $b_{\xi}=b_{o} \frac{l-\xi}{l}, b_{x}=b_{o} \frac{l-x}{l}$. The inertia moment of the curent sectional area can be calculated with the relation :

$$
\begin{equation*}
I_{z}(\xi)=b_{\xi} \frac{h^{3}}{12}=b_{o} \frac{l-\xi}{l} \cdot \frac{h^{3}}{12} \tag{1}
\end{equation*}
$$

The bending moment in the current section of the beam can be expressed by reducing all the moments from the left side:

$$
M_{z}(\xi)= \begin{cases}F \frac{l-x}{l} \cdot \xi, & \xi \in[0, x)  \tag{2}\\ F \frac{x}{l} \cdot(l-\xi), & \xi \in[x, l]\end{cases}
$$

The second order differential equation that describes the bending of the elastic curve of the beam can be written under the form:

$$
\frac{d^{2} v}{d \xi^{2}}=-\frac{M_{z}(\xi)}{E I_{z}(\xi)}= \begin{cases}-\frac{12 F}{E b_{o} h^{3}} \cdot \frac{\xi}{l-\xi}(l-x), & \xi \in[0, x)  \tag{3}\\ -\frac{12 F}{E b_{o} h^{3}} \cdot x, & \xi \in[x, l]\end{cases}
$$

In order to obtain the expressions of the slope and deflection the above differential equation has to be integrated step by step in respect with the current variable $\xi$ :

$$
\begin{gather*}
\varphi(\xi)=\frac{d v}{d \xi}= \begin{cases}\frac{12 F(l-x)}{E b_{o} h^{3}}\left[\xi+l \cdot \ln (l-\xi)+C_{1}\right], & \xi \in[0, x) \\
-\frac{12 F}{E b_{o} h^{3}} \cdot x\left(\xi+C_{3}\right), & \xi \in[x, l]\end{cases}  \tag{4}\\
v(\xi)= \begin{cases}\frac{12 F}{E b_{o} h^{3}}(l-x)\left[\frac{\xi^{2}}{2}-l \cdot(l-\xi) \cdot \ln (l-\xi)-l \cdot \xi+C_{1} \cdot \xi+C_{2}\right], & \xi \in[0, x) \\
-\frac{12 F}{E b_{o} h^{3}} \cdot x \cdot\left(\frac{\xi^{2}}{2}+C_{3} \cdot \xi+C_{4}\right), & \xi \in[x, l]\end{cases} \tag{5}
\end{gather*}
$$

The integration constants $\left(C_{1}, C_{2}, C_{3}, C_{4}\right)$ can be find from the limit conditions that are function of the edge supports of the beam.

For a simply supported beam such that presented in figure 1 the limit conditions are :

$$
\xi=0 \Rightarrow v(0)=0 \quad \text { a); } \quad \xi=l \Rightarrow v(l)=0 \quad \text { b) } ; \quad \xi=x \Rightarrow\left\{\begin{array}{l}
\varphi_{x_{-}}=\varphi_{x_{+}}  \tag{6}\\
v_{x_{-}}=v_{x_{+}}
\end{array} \quad\right. \text { c) }
$$

Replacing (4) and (5) relations in (6) the following algebric system of equations is obtained :

$$
\begin{align*}
& -l^{2} \cdot \ln l+C_{2}=0 \\
& \frac{l^{2}}{2}+C_{3} \cdot l+C_{4}=0 \\
& (l-x) \cdot\left[\frac{x^{2}}{2}-l \cdot(l-x) \cdot \ln (l-x)-l \cdot x+C_{1} \cdot x+C_{2}\right]=-x\left(\frac{x^{2}}{2}+C_{3} \cdot x+C_{4}\right) \\
& (l-x) \cdot\left[x+l \cdot \ln (l-x)+C_{1}\right]=-x \cdot\left(x+C_{3}\right)
\end{align*}
$$

Solving the above system of equations the following solutions are obtained:

$$
\begin{array}{ll}
C_{1}=\frac{x}{2}-l \cdot \ln l \\
C_{2}=l^{2} \cdot \ln l \\
C_{3}=\frac{x-3 l}{2}+\frac{l(l-x)}{x} \ln \frac{l}{l-x} & \text { a) }  \tag{8}\\
C_{4}=\frac{2 l^{2}-x \cdot l}{2}-\frac{l^{2}(l-x)}{x} \ln \frac{l}{l-x}
\end{array}
$$

Using the (8) relations the expressions of the slope and deflection can be written under the form:

$$
\begin{align*}
& \varphi(\xi, x)=\left\{\begin{array}{l}
\frac{12 F}{E b_{o} h^{3}}(l-x)\left[\xi+l \cdot \ln (l-\xi)+\frac{x}{2}-l \cdot \ln l\right], \quad \xi \in[0, x) \\
-\frac{12 F}{E b_{o} h^{3}}(x)\left[\begin{array}{l}
\left.\xi+\frac{x-3 l}{2}+\frac{l \cdot(l-x)}{x} \ln \frac{l}{l-x}\right], \quad \xi \in[x, l]
\end{array}\right. \\
v(\xi, x)=\left\{\begin{array}{l}
\frac{12 F}{E b_{o} h^{3}}(l-x)\left[\begin{array}{l}
\left.\frac{\xi^{2}}{2}-l \cdot(l-\xi) \cdot \ln (l-\xi)-l \cdot \xi+\frac{x \cdot \xi}{2}-\right], \\
-l \cdot \xi \cdot \ln l+l^{2} \ln l
\end{array}\right] \in[0, x) \\
-\frac{12 F}{E b_{o} h^{3}} x\left[\begin{array}{l}
\frac{\xi^{2}}{2}+\frac{x-3 l}{2} \xi+\frac{l \cdot \xi \cdot(l-x)}{x} \ln \frac{l}{l-x}+ \\
+\frac{2 l^{2}-x \cdot l}{2}-\frac{l^{2}(l-x)}{x} \cdot \ln \frac{l}{l-x}
\end{array}\right],
\end{array} \quad \xi \in[x, l]\right.
\end{array}\right. \tag{9}
\end{align*}
$$

The (9) and (10) expressions allow the calculation of the displacements of the beam in any section and for any position of the concentrated force along the $x$ axis.

Ussualy the displacement has a maximum in the neibourhoud of the action of the concentrated force. In order to find the which section coresponds to the maximum displacement in the same point where the concentrated force acts is necessary to impose the analitical condition :

$$
\begin{equation*}
\varphi_{x_{-}}=\varphi_{x_{+}} \tag{11}
\end{equation*}
$$

Replacing (9) in (11) the following equality is obtained:

$$
\begin{equation*}
(l-x)\left[x+l \cdot \ln (l-x)+\frac{x}{2}-l \cdot \ln l\right]=-x \cdot\left[x+\frac{x-3 l}{2}+\frac{l(l-x)}{x} \ln \frac{l}{l-x}\right]=0 \tag{12}
\end{equation*}
$$

The solutionof the (12) equations (if exists) represents the abscisa of the point where the deflection of the beam has a maximum in the same point where the concentrated force acts.
In order to find the nummerical solution of the (13) equations a specialised prgram has been used.

## A Numerical Example

Finding the section with maximum deflection is important especially for impact phenomena. The beam presented in figure (1) is considered to have the following geometrical data : the length $l=1 \mathrm{~m}$, the origin depth of the transversal section $b_{o}=100 \mathrm{~mm}$, the thickness $h=10 \mathrm{~mm}$, the longitudinal elasticity modulus of the steel $E=210000 \mathrm{~N} / \mathrm{mm}^{2}$ and the concentrated force $F$ $=1 \mathrm{~N}$.

The first step in finding the maximum deflection of the beam is to find the solution of (12) equations. For this respect a nummerical program has been used and the grapphical representation of the functions defined in (12) equality is presented in figure 2 (in the neibourhood of the solution).


Fig. 2. Finding the numerical solution

As it can be noticed from the fig. 2 the both equations from (12) have the same solutions $x_{o}=$ 582.8 mm . This means that in the point located at $x_{o}$ from the left edge of the beam is to be avoided as a action point for the concentrated force.

The graphical representation of the slope and deflection are presented in figures 3 and 4 for the both intervals of the beam (A-C and C-B). It can be noticed from figure 4 that the maximum deflection of the beam reached the value $v_{\max }=0.263 \mathrm{~mm}$.
$\varphi[\mathrm{rad}]$


Fig. 3a. The slope of the beam for $\mathrm{x}<\mathrm{x}_{0}$


Fig. 3b. The slope of the beam for $x>x_{0}$
Analysing the figures presented above it can be noticed that the slope in the origin of axis (in A) is positive and in the right ende is negative. The slope are zero (on the both intervals) at the $x_{o}$ abscisa, that means that in this point the deflection is maximum.

The figures represented bellow show the variation of the deflection of the beam, for both intervals (A-C and C-B). The maximum value of the deflection (when the concentrated force acts in the same point) is reached in the $x_{0}$ abscisa and has the value $v_{\text {max }}=0,263 \mathrm{~mm}$.


Fig. 4. The deflection of the beam

## Conclusions

In the paper is presented a way of finding the most dangerous section for the impact phenomena for a beam with variable cross sectional area loaded with an external concentrated force.
It was founded the section where the concentrated force produce the maximum deflection in the same point where the force acts. There are also presented the analytical and graphical solution for slope and deflection. The procedure presented in the paper can be used for some other restrictions as well.

## References

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## Aspecte privind deformarea barelor cu secțiune variabilă

## Rezumat

În lucrare se prezintă o metodologie pentru determinarea deplasarii maxime la o bara cu seciunea variabila. Este analizat cazul defavoravil cand deplasarea maxima se obtine chiar in punctul in care actioneaza forta concentrata, deoarece o astfel de situatie trebuie evitata, in special cand apar si fenomene de impact transversal. Rezultatele obtinute sunt analizate pe un exemplu de calcul.

