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A Time Domain Method for Computing the Response of the Input-Output Linear Discrete Systems

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Abstract

The paper presents a unitary method for computing the forced response in the time domain of an input-output SISO linear discrete invariant system described by a difference equation. The main idea is to use the superposition principle in order to replace the primary difference equation with an equivalent suitable model of two equations, which reflects an indirect input-output transfer by means of a new internal variable. Using this method, we may compute the system response in two steps, by choosing all the initial conditions equal to zero.

Key words: *input-output model*, *primary/secondary model*, *superposition principle*, *difference equation.*

Introduction

Computing the time domain response of input-output linear discrete systems to a given input is done using the assumption that the system was at steady-state until the initial time $t_0 = 0$, with all the input and output variables equal to zero (original type variables).

Input-Output Model of Discrete Systems

The mathematical model of a SISO and time-invariant linear discrete system, with sampling period $T = 1$, has the following primary form:

$$a_0 y(t) + a_1 y(t-1) + \cdots + a_n y(t-n) = b_0 u(t) + b_1 u(t-1) + \cdots + b_r u(t-r), \qquad (1)$$

where a_i and b_i coefficients are real constants, and $a_0 \neq 0$. Replacing variable t with integer variable $k \in \mathbf{Z}$, the primary model can be written under the simplified form

$$a_0 y_k + a_1 y_{k-1} + \cdots + a_n y_{k-n} = b_0 u_k + b_1 u_{k-1} + \cdots + b_r u_{k-r} \,. \qquad (2)$$

The system described by the equivalent equations (1) and (2) is called *proper* and has the order equal to $\max\{n, r\}$. If $b_0 = 0$ then the system is called *strictly proper*, and if $b_0 \neq 0$ then the system is called *semi-proper*. The response of a strictly proper system to any given input function is delayed with one or many sampling periods T. The response of a semi-proper system contains a component which follows instantaneously the input function variation.

For $a_0 + a_1 + ... + a_n \neq 0$ and $b_0 + b_1 + \cdots + b_r \neq 0$, the system is of *proportional type*. The stationary model of a proportional system has the form $y = K\,u$, with the static gain

$$K = \frac{b_0 + b_1 + \cdots + b_r}{a_0 + a_1 + ... + a_n}. \tag{3}$$

For $a_0 + a_1 + ... + a_n = 0$ and $b_0 + b_1 + \cdots + b_r \neq 0$, the system is of *integral type*. A *purely integral* system has the following model

$$y(t) - y(t-1) = b_0 u(t).$$

For $a_0 + a_1 + ... + a_n \neq 0$ and $b_0 + b_1 + \cdots + b_r = 0$, the system is of *derivative type*. A *purely derivative* system has the following model

$$a_0 y(t) = u(t) - u(t-1).$$

According to the superposition principle, the primary model (1) with $a_0 = 1$ is equivalent with the following *secondary models*

$$\begin{cases} w(t) + a_1 w(t-1) + ... + a_n w(t-n) = u(t) \\ y(t) = b_0 w(t) + b_1 w(t-1) + ... + b_r w(t-r) \end{cases}, \tag{4}$$

respectively

$$\begin{cases} w(t) + a_1 w(t-1) + ... + a_n w(t-n) = u(t-n) \\ y(t) = b_0 w(t+n) + b_1 w(t+n-1) + ... + b_r w(t+n-r) \end{cases}. \tag{5}$$

In both cases, the input-output transfer is made indirectly, through variable w.

Discrete System Response Computation

Usually, the secondary models (4) or (5) are used for computation of the analytical or numerical forced response of a discrete system to a given input. The variables u, y and w from the secondary models are all of *original* type; that is, they are equal to zero for any negative value of their argument.

The *numerical computation* of the intermediary variable w for $t \geq 0$ to a given original input u can be made from (4) using the recurrence relation:

$$w_i = u_i - a_1 w_{i-1} \cdot 1^o(i-1) - \cdots - a_n w_{i-n} \cdot 1^o(i-n), \quad i = 0, 1, 2, \cdots. \tag{6}$$

where $1^o(t)$ is the unit step discrete function (Fig. 1):

$$1^o(t) = \begin{cases} 0, & t < 0 \\ 1, & t = 0, 1, 2, \cdots \end{cases}$$

Furthermore, the numerical values of the response $y(t)$ are given by the relations

$$y_i = b_0 w_i + b_1 w_{i-1} \cdot 1^o(i-1) + \cdots + b_r w_{i-r} \cdot 1^o(i-r), \quad i = 0, 1, 2, \cdots \tag{7}$$

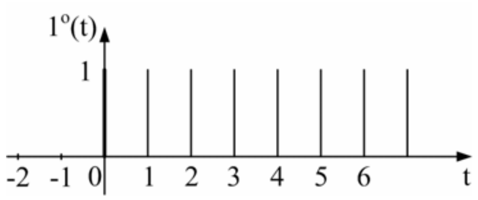

Fig.1. Unit step discrete function

Because $w_j = 0$ for any $j < 0$, for $i = 0, 1, 2, \cdots$ relation (6) can be expressed as

$$w_i = u_i - a_1 w_{i-1} - \cdots - a_n w_{i-n}\,, \tag{8}$$

which is equivalent to

$$w_i = u_i - \sum_{j=1}^{\min\{i,n\}} a_j w_{i-j}\,. \tag{9}$$

Similarly, for $i = 0, 1, 2, \cdots$ relation (7) can be expressed as

$$y_i = b_0 w_i + b_1 w_{i-1} + \cdots + b_r w_{i-r}\,, \tag{10}$$

or

$$y_i = \sum_{j=0}^{\min\{i,r\}} a_j w_{i-j}\,. \tag{11}$$

The analytical response $y(t)$ of the system (1) to a given original analytical input $u = u_0(t)$ can be better determined by the secondary models (4) and (5). We will present next a method based on the model (5), which allows us to take all the initial conditions equal to zero:

$$w(0) = w(1) = \cdots = w(n-1) = 0\,. \tag{12}$$

The response $w(t)$ to a given analytical input function $u = f(t) \cdot 1^{\circ}(t)$ can be expressed as

$$w(t) = w_{\mathrm{om}}(t) + w_{\mathrm{p}}(t)\,,$$

where $w_{\mathrm{om}}(t)$ is the solution of the difference homogeneous equation

$$w(t) + a_1 w(t–1) + ... + a_n w(t–n) = 0\,,$$

and $w_{\mathrm{p}}(t)$ is a particular solution of the difference non-homogeneous equation

$$w(t) + a_1 w(t–1) + ... + a_n w(t–n) = f(t-n)\,. \tag{13}$$

Note that the analytical solutions $w_{\mathrm{om}}(t)$ and $w_{\mathrm{p}}(t)$ are usually valid only for $t \geq n$. The homogeneous equation solution has the form

$$w_{\mathrm{om}}(t) = C_1 z_1^t + C_2 z_2^t + \cdots + C_n z_n^t\;, \tag{14}$$

where $z_1\,, z_2\,, \cdots, z_n$ are (distinct) roots of the characteristic equation

$$z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0\,, \tag{15}$$

and $C_1, C_2, \cdots, C_n$ are real or complex-conjugate constants. If the roots z_1 and z_2 are real and equal, then the sum $C_1 z_1^t + C_2 z_2^t$ has to be replaced by $(C_1 t + C_2) z_1^t$. If z_1 and z_2 are complex-conjugate, that is $z_{1,2} = \rho(\cos\alpha \pm j\sin\alpha)$, then the sum $C_1 z_1^t + C_2 z_2^t$ with C_1 and C_2 complex-conjugate constants may be replaced with the real expression $\rho^t (C_1 \cos\alpha t + C_2 \sin\alpha t)$, where C_1 and C_2 real constants.

The constants C_1, C_2, $\cdots$, C_n are determined so that the general solution $w(t) = w_{\text{om}}(t) + w_{\text{p}}(t)$, initially valid for $t \ge n$, verifies the null initial conditions (12) at moments $n-1$, $n-2$, ..., 0. Thus, the solution $w(t)$ becomes valid for any $t \ge 0$.

The particular solution $w_{\text{p}}(t)$ of the equation (13) usually has a similar form with the one of the input function $f(t)$. Thus,

- for unit impulse input, $u(t) = \delta^0(t)$ - fig. 2

$$w_{\text{p}}(t) = 0, \text{ for } t \ge n+1;$$

- for unit step input

$$w_{\text{p}}(t) = \frac{1}{1 + a_1 + a_2 + \ldots + a_n}, \text{ for } t \ge n;$$

- for unit ramp input, $u(t) = t \cdot 1^0(t)$

$$w_{\text{p}}(t) = \frac{a_1 + 2a_2 + \ldots + na_n}{(1 + a_1 + a_2 + \ldots + a_n)^2} + \frac{t-n}{1 + a_1 + a_2 + \ldots + a_n}, \text{ for } t \ge n;$$

- for exponential input, $u(t) = a^t \cdot 1^0(t)$

$$w_{\text{p}}(t) = \frac{a^{t-n}}{1 + a_1 a^{-1} + a_2 a^{-2} + \ldots + a_n a^{-n}}, \text{ for } t \ge n.$$

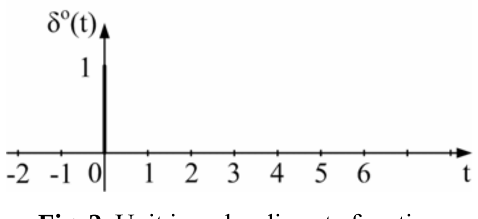

Fig. 2. Unit impulse discrete function

If the characteristic equation (15) has the distinct roots $z_1, z_2, \cdots, z_n$, and $1 + a_1 + a_2 + \ldots + a_n \ne 0$, then the solution $w(t)$ of the difference equation (13) to a unit step input $u = 1^0(t)$, can be expressed for $t \ge 0$ as

$$w(t) = \frac{1}{1 + a_1 + a_2 + \ldots + a_n} + C_1 z_1^t + C_2 z_2^t + \cdots + C_n z_n^t \,,$$

where $C_1, C_2, \cdots, C_n$ are real or complex constants (C_i is real/complex if z_i is real/complex).

Taking into account the output equation

$$y(t) = b_0 w(t+n) + b_1 w(t+n-1) + \ldots + b_r w(t+n-r)\,,$$

the step response $h(t)$ of the system for $t \geq \min\{0, r-n\}$ is of the form

$$h(t) = \frac{b_0 + b_1 + \cdots + b_r}{1 + a_1 + a_2 + \ldots + a_n} + D_1 z_1^t + D_2 z_2^t + \cdots + D_n z_n^t, \qquad (16)$$

where $D_1, D_2, \cdots, D_n$ are real or complex constants (D_i is real/complex if z_i is real/complex). If the roots z_1 and z_2 are complex-conjugate, that is

$$z_{1,2} = \rho(\cos\alpha \pm j\sin\alpha)\,,$$

then the sum $D_1 z_1^t + D_2 z_2^t$ can be expressed as $\rho^t(E_1 \cos\alpha t + E_2 \sin\alpha t)$, where E_1 and E_2 are real constants. From the expression (16), we see that the unit step response $h(t)$ is bounded when all the characteristic equation roots $z_1, z_2, \cdots, z_n$ has their modulus less than 1, that is, they lie inside the unit circle centered at the origin in the complex-plane. This result is also valid when the characteristic equation has multiple roots.

The response $g(t)$ to a unit impulse input, $u = \delta^{\mathrm{o}}(t)$, is called impulse response function or, sometimes, weighting response function, while the response $h(t)$ to a unit step input, $u = 1^{\mathrm{o}}(t)$, is called step response function or, sometimes, indicial response function. By the superposition principle, from the relations

$$\delta^{\mathrm{o}}(t) = 1^{\mathrm{o}}(t) - 1^{\mathrm{o}}(t-1)\,, \; 1^{\mathrm{o}}(t) = \delta^{\mathrm{o}}(t) + \delta^{\mathrm{o}}(t-1) + \cdots + \delta^{\mathrm{o}}(0)\,, \qquad (17)$$

it follows that between weighting response function $g(t)$ and indicial response function $h(t)$ there are the following relations

$$g(t) = h(t) - h(t-1)\,, \; h(t) = g(t) + g(t-1) + \cdots + g(0)\,. \qquad (18)$$

If the weighting response function $g(t)$ is known, then the system response to a given original arbitrary input $u(t)$ can be computed using the *convolution relation*:

$$y(t) = g(t)u(0) + g(t-1)u(1) + \cdots + g(0)u(t) = \sum_{i=0}^{t} g(t-i)u(i)\,. \qquad (19)$$

Relation (19) results immediately from the superposition principle, taking into account the fact that

$$u(t) = u(0)\cdot 1^0(t) + u(1)\cdot 1^0(t-1) + \cdots + u(t)\cdot 1^0(0) = \sum_{i=0}^{t} u(i)\cdot 1^0(t-i)\,.$$

Remark. The computing procedure is similar when the secondary model is used under the form (4) or the form

$$\begin{cases} w(t)+a_1 w(t{-}1)+\ldots+a_n w(t{-}n)=u(t-1) \\ y(t)=b_0 w(t+1)+b_1 w(t)+\ldots+b_r w(t{+}1-r) \end{cases}.$$

In these cases, the initial conditions $w(0)$, $w(1)$, … , $w(n-1)$ are obtained directly from difference equation in w by replacing successively t with 0, 1, …, $n-1$.

Conclusions

The presented method is used to determine the time domain response of an input-output discrete linear system to any given original input function. Using the equivalent secondary model (5), two advantages are accomplished: (a) a relevant simplification of the computing operations, by choosing all initial conditions equal to zero; (b) a better organization, in two steps, of the computing operations.

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Metodă de calcul în domeniul timpului a răspunsului sistemelor liniare discrete de tip intrare-ieşire

Rezumat

În lucrare este prezentată o metodă unitară de calcul în domeniul timpului a răspunsului sistemelor liniare discrete monovariabile şi invariante de tip intrare-ieşire, descrise prin ecuaţii cu diferenţe. Ideea centrală este utilizarea principiului superpoziţiei în vederea înlocuirii ecuaţiei primare cu un model echivalent de două ecuaţii, numit model secundar, care realizează un transfer indirect intrare-ieşire, prin intermediul unei variabile interne. Utilizând această metodă putem calcula răspunsul sistemului în două etape, prin alegerea tuturor condiţiilor iniţiale nule.