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A Time Domain Method for Computing the Forced Response of an Input-Output Linear Continuous System

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Abstract

The paper presents a unitary method for computing the forced response in the time domain of a SISO time-invariant linear continuous system of input-output type, described by a differential equation. The proposed procedure uses the superposition principle in order to replace the primary differential equation with a suitable equivalent model of two equations, called secondary model, which reflects an indirect input-output transfer by means of a new internal variable. Since the secondary model does not contain derivatives of the input variable, we can determine the system response to any non-differentiable original input function, such as the unit step function. Moreover, all the initial conditions are equal to zero in our proposed procedure.

Key words: *input-output model, primary model, secondary model, non-differentiable input functions*

Introduction

Usually, the time domain analysis of the input-output linear continuous systems is done under the assumption that the system was at steady state until the initial time $t_0 = 0$, with all the input and output variables equal to zero (original type variables). The system response $y(t)$ to a given original input $u(t)$ is a forced response.

Input-Output Model of Continuous Systems

The mathematical model of a SISO linear continuous and time-invariant system of n^{th} order, has the following *primary form*:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 \dot{y} + a_0 y = b_r u^{(r)} + b_{r-1} u^{(r-1)} + \cdots + b_1 \dot{u} + b_0 u, \tag{1}$$

where a_i and b_i are real constants and $a_n \neq 0$.

Conventionally, the input variable u and the output variable y represent variations of the real system physical variables compared to their initial values. As a result, if the system is at steady state before the initial time $t_0 = 0$, then both system variables are zero for $t < 0$.

If $r \le n$, then the system is called *proper* (*strictly proper* for $r < n$, and *semi-proper* for $r = n$). If $r > n$, the system is *improper*.

An improper system doesn't satisfy the causality principle. All physical systems are proper since they satisfy this principle.

A strictly proper system clearly satisfies the causality principle, since the input-output transfer is strictly delayed.

A semi-proper system satisfies the causality principle just on the line, since its output has a component which follows instantaneously any input variation.

Canceling all derivatives of the input and output variables, from the dynamic model (1) we obtain the *stationary model* $y = Ku$, with the static gain $K = \dfrac{b_0}{a_0}$.

If $a_0 \ne 0$ and $b_0 \ne 0$, then the slope of the static characteristic is finite and nonzero; such a system is of *proportional* type. Most of the physical systems are of proportional type. The step response of a proportional and stable system settles to a finite and nonzero final value.

If $a_0 = 0$ and $b_0 \ne 0$, then the system is of *integral* type. An integral system can be at steady state only if the input u is equal to zero. A *purely integral* system has the model $a_1\dot{y} = b_0 u$, which is equivalent to

$$y = \frac{b_0}{a_1}\int_0^t u\,\mathrm{d}t.$$

The step response of a purely integral system is a ramp response (with constant slope). Generally, the step response of an integral stable system approaches asymptotically an inclined straight line, being a delayed ramp response. An integral system is a "persistent" system, because the output y doesn't settle if the input u is nonzero.

If $a_0 \ne 0$ and $b_0 = 0$, then the system is of *derivative* type. A derivative system has the stationary model $y = 0$. Because the value of the output variable y is zero at steady state, the step response of a derivative stable system approaches zero. Due to the impulse form of the step response (which starts from zero and goes towards zero), a derivative system is a "lead" system.

The improper model

$$y = \frac{b_1}{a_0} \cdot \frac{\mathrm{d}u}{\mathrm{d}t}$$

describes a *purely derivative* system, and the model

$$a_1 \frac{\mathrm{d}y}{\mathrm{d}t} + a_0 y = b_1 \frac{\mathrm{d}u}{\mathrm{d}t}, \quad a_1 \ne 0$$

describes a semi-proper derivative system.

According to the superposition principle, the primary model (1) is equivalent to the following *secondary* model:

$$\begin{cases} a_n w^{(n)} + a_{n-1} w^{(n-1)} + \cdots + a_1 \dot{w} + a_0 w = u \\ y = b_r w^{(r)} + b_{r-1} w^{(r-1)} + \cdots + b_1 \dot{w} + b_0 w \end{cases} . \tag{2}$$

By the superposition principle, the effect of a sum of causes is equal to the sum of the effects of the each cause, and, moreover, a multiplied/differentiated cause yields a multiplied/differentiated effect. The output y from the primary model (1) is the effect of a weighted sum of $r+1$ derivatives of the cause u, while the output y from the secondary model (2) is the weighted sum of $r+1$ derivatives of the effect w of the cause u. On the other hand, we can check that formally equation (1) becomes identity by replacing the variables u and y from equations (2) in equation (1).

For the secondary model, the input-output transfer is performed indirectly, by the aid of the variable w.

Because the secondary model doesn't contain derivatives of the input u, we can also use this model for *non-differentiable* or *discontinuous* input variables.

Continuous System Response Computation

Usually, the secondary model (2) is used for computation of the analytical response of a linear system to a given differentiable or even non-differentiable original input.

According to the secondary model, the system response for $t \geq 0$ to any input original analytical function, $u = f(t) \cdot 1(t)$, is given by the relation

$$y = b_r w^{(r)} + b_{r-1} w^{(r-1)} + \cdots + b_1 \dot{w} + b_0 w, \tag{3}$$

where $w(t)$ is the solution of the differential equation

$$a_n w^{(n)} + a_{n-1} w^{(n-1)} + \cdots + a_1 \dot{w} + a_0 w = f(t), \tag{4}$$

for null initial conditions:

$$w(0+) = \dot{w}(0+) = \cdots = w^{(n-1)}(0+) = 0 . \tag{5}$$

The n null initial conditions express the continuity of the functions $w(t)$, $w'(t)$, ... , $w^{(n-1)}(t)$ at the initial time $t = 0$. Indeed, since the derivative of a function is equal to the function changing rate, a step variation at $t = 0$ of the function $w^{(i)}(t)$, $0 \leq i \leq n-1$, implies an infinite variation of its derivative $w^{(i+1)}(t)$, and this result contradicts equation (4).

The condition that equation (4) is satisfied at $t = 0+$ yields

$$w^{(n)}(0+) = \frac{f(0+)}{a_n} . \tag{6}$$

From equation (3) and the initial conditions (5) and (6), it follows that the response $y(t)$ of the system (1) satisfies the following initial conditions:

$$y(0+) = \dot{y}(0+) = \cdots = y^{(n-r+1)}(0+) = 0, \quad y^{(n-r)}(0+) = \frac{b_r f(0+)}{a_n} . \tag{7}$$

The initial conditions theorem. *The response of the continuous linear system* (1)*, with* $a_n \neq 0$ *and* $b_r \neq 0$ *, to any finite original input function has at least* $n-r$ *null initial conditions.*

For input original functions which are discontinuous in origin, with the form $u = f(t)\cdot 1(t)$, $f(0+) \neq 0$, the system response is characterized by exactly $n-r$ null initial conditions. In particularly, for a unit step input, $u = 1(t)$, the first non-zero initial condition of the response is $y^{(n-r)}(0+) = \dfrac{b_r}{a_n} \neq 0$.

The system response is characterized by at least $n-r+1$ null initial conditions for continuous original input functions, and at least $n-r+2$ null initial conditions for differentiable original input functions.

The response to a unit step input,

$$u = 1(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases},$$

is called *indicial response function* or *unit step response function*, and the response to a Dirac impulse input, $u = \delta_0(t)$, is called *weighting response function* or *Dirac impulse response function*. Next, we denote the indicial function by $h(t)$, and the weighting function by $g(t)$. Both of these functions are original functions (equal to zero for $t < 0$).

For a unit step input $u = 1(t)$, the solution of the differential equation (4), with $a_0 \neq 0$, can be expressed as

$$w(t) = \frac{1}{a_0} + C_1 \mathrm{e}^{s_1 t} + C_2 \mathrm{e}^{s_2 t} + \cdots + C_n \mathrm{e}^{s_n t}, \quad t \geq 0,$$

where $s_1, s_2, \cdots, s_n$ are the roots of the characteristic equation

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0, \tag{8}$$

and $C_1, C_2, \cdots, C_n$ are real or complex constants (C_i is real/complex when s_i is real/complex).

If $s_1 = s_2$, then the sum $C_1 \mathrm{e}^{s_1 t} + C_2 \mathrm{e}^{s_2 t}$ has to be replaced by $(C_1 t + C_2)\mathrm{e}^{s_1 t}$. If the roots s_1 and s_2 are complex-conjugate, namely $s_{1,2} = a \pm jb$, then the sum $C_1 \mathrm{e}^{s_1 t} + C_2 \mathrm{e}^{s_2 t}$, with C_1 and C_2 complex-conjugate constants, may be replaced with the expression $\mathrm{e}^{at}(C_1 \sin bt + C_2 \cos bt)$, with C_1 and C_2 real constants. The constants $C_1, C_2, \cdots, C_n$ are computed from the null initial conditions (5).

Taking into account relation (3), in the case when the characteristic equation (8) has distinct roots, the unit step response $h(t)$ for $t \geq 0$ has the form

$$h(t) = \frac{b_0}{a_0} + D_1 \mathrm{e}^{s_1 t} + D_2 \mathrm{e}^{s_2 t} + \cdots + D_n \mathrm{e}^{s_n t},$$

where $D_1, D_2, \cdots, D_n$ are real or complex constants (D_i is real/complex when s_i is real/complex). If the roots s_1 and s_2 are complex-conjugate, $s_{1,2} = a \pm jb$, then the sum $D_1 e^{s_1 t} + D_2 e^{s_2 t}$ can be expressed as $e^{at}(E_1 \sin bt + E_2 \cos bt)$, where E_1 and E_2 are real constants. From the unit step response $h(t)$, we see that this is bounded when all the characteristic equation roots $s_1, s_2, \cdots, s_n$ have their real part negative. This result is valid also when the characteristic equation has multiple roots.

By the superposition principle, it follows that between the unit step response $h(t)$ and the Dirac impulse response $g(t)$ there are the following relations:

$$h(t) = \int_{0-}^{t} g(\tau)\mathrm{d}\tau, \quad g(t) = \frac{\mathrm{d}h(t)}{\mathrm{d}t}. \tag{9}$$

According to the last relation, the weighting function $g(t)$ is equal to the distributional derivative (in a generalized mode) of the indicial function $h(t)$. This means that if $h(t)$ is discontinuous at $t = 0$, then $g(t)$ is a tempered distribution given by

$$g(t) = \dot{h}(t) + h(0+)\delta_0(t), \tag{10}$$

where $\dot{h}(t)$ is the classical derivative (assumed to be 0 at $t = 0$).

The weighting function $g(t)$ has a significant role in the theoretical study of linear dynamic systems, due to the fact that it allows a relatively brief expression of the response to an original input function.

The convolution theorem. *If the weighting function $g(t)$ of a continuous linear system is given, then the response $y(t)$ to a given original input $u(t)$ can be expressed by the convolution relation*:

$$y(t) = \int_0^t g(t-\tau)u(\tau)\mathrm{d}\tau. \tag{11}$$

The convolution relation (11) is a direct consequence of the superposition principle. Indeed, since the Dirac impulse input $\delta_0(t)$ yields the response $g(t)$, then to the input $u(t)$, which can be written as $\int_0^t \delta_0(t-\tau)u(\tau)\mathrm{d}\tau$, yields the response $\int_0^t g(t-\tau)u(\tau)\mathrm{d}\tau$.

Remarks.

1⁰. With regard to the unit step response $h(t)$ of the system (1), we make the following observations.

a) for $r = n$ (the case of a semi-proper system, which satisfy just on the line the causality principle), the unit step response is discontinuous at $t = 0$, because

$$h(0+) = b_n / a_n \neq 0;$$

b) for $r = n-1$ ($b_n = 0$, $b_{n-1} \neq 0$), the unit step response is continuous, but non-differentiable at $t = 0$, because

$$h(0+) = 0\,,\ \ \dot{h}(0+) = b_{n-1}/a_n \neq 0\,;$$

c) for $r = n-2$ ($b_n = b_{n-1} = 0\,, b_{n-2} \neq 0$), the unit step response is continuous and simply differentiable at $t = 0$ (tangent to the t - axis), because

$$h(0+) = \dot{h}(0+) = 0\,,\quad \ddot{h}(0+) = b_{n-2}/a_n \neq 0\,.$$

2^0. For a serial, parallel, or feedback connected system (fig. 1, 2, 3), the response to a given original input is usually determined using the model of the entire system, obtained from the sub-system models by eliminating all intermediary variables (of v variable to the serial connection, of v_1 and v_2 variables to the parallel connection, of e and v variables to the feedback connection).

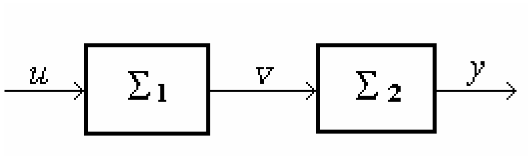

Fig. 1. Serial connection

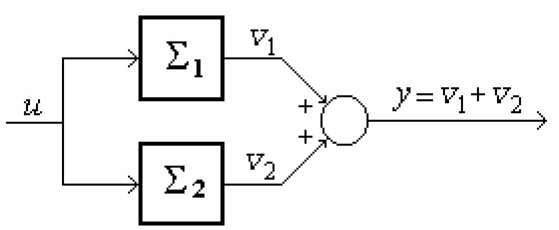

Fig. 2. Parallel connection

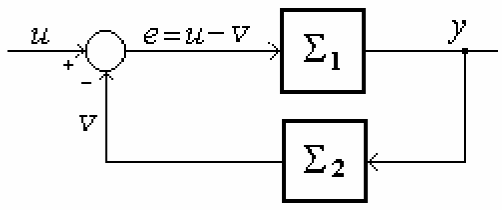

Fig. 3. Feedback connection

For open loop connections (serial or parallel), the response can be also determined using a step by step procedure; that is, to calculate successively the response of each sub-system to the already computed input.

Conclusions

The presented method allows computing the response in the time domain to any continuous or discontinuous original input function for an input-output linear continuous system. Using the secondary model (2), three advantages are accomplished: (a) a relevant simplification of the calculation, by choosing all the initial conditions equal to zero; (b) an extension of the proposed computing method to non-differentiable or even discontinuous input functions, by canceling all the input derivatives of the model; (c) a more rational arrangement, in two steps, of the computing operations.

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Metodă de calcul în domeniul timpului a răspunsului sistemelor liniare continue de tip intrare-ieşire

Rezumat

In lucrare este prezentată o metodă unitară de calcul în domeniul timpului a răspunsului sistemelor liniare continue monovariabile şi invariante, descrise prin ecuații diferențiale de tip intrare-ieşire. Ideea principală constă în utilizarea principiului superpoziției pentru înlocuirea ecuației primare cu un model echivalent de două ecuații, care realizează un transfer indirect intrare-ieşire, prin intermediul unei variabile interne. Deoarece modelul echivalent nu conține derivate ale variabilei de intrare, metoda poate fi aplicată în calculul răspunsului la intrări nederivabile sau chiar discontinue, cum este intrarea de tip treaptă. In plus, metoda propusă beneficiază de avantajul considerării tuturor condițiilor inițiale nule.