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The Graetz Problem for Power-law Fluids Flow with Neumann Boundary Conditions

Tudor Boacă

Petroleum and Gas University of Ploieşti, Bd. Bucureşti 39, Ploieşti, România
e-mail: tboaca@upg-ploiesti.ro

Abstract

This paper considers the Graetz problem in the flow of power-law fluid through a tube of circular cross section, with Neumann boundary conditions. The solution of the problem is obtained by a series expansion about the complete eigenfunctions system of a Sturm-Liouville problem. Eigennfunctions and eigenvalues of this Sturm-Liouville problem are obtained by Galerkin's method.

Keyword: *Graetz, power law fluid, eigenfunction, Galerkin's method*

Introduction

The Graetz problem [1] describes the temperature (or concentration) field in fully-developed laminar flow in a circular tube where the wall temperature (or concentration) profile is a step function [2]. This problem and its various extensions were studied in numerous articles.

The analytical results of Papoutsakis, Ramkrishna and Lim [3], [4] obtained for the extended Graetz problem with axial conduction, are used by Papoutsakis [5] for the determination of the behaviour of Nusselt number very close to the entrance of the termal region.
Weigand showed how the method of Papoutsakis and all [4] can be adapted for solving the extended turbulent Graetz problem for constant wall temperature [6] and for piecewise constant wall heat flux [7] for pipe and channel flows.
A numerical evaluation of the Graetz series and a comparison with Lévêque solution [8] are given in [9]. The Bessel series and power series expansions of Kummer function are given here.

In some previous articles [10], [11], [12], [13], we proposed an approximate method for the determination of viscous dissipation in power-law fluid flow through tubes of circular cross section. The method used in these articles will be used for solving the Graetz problem for power-law fluids flow with Neumann boundary conditions.

Now we will consider the flow of power-law fluid through a tube of circular cross section with Neumann boundary conditions (constant flux wall). At the entrance of tube the temperature of fluid is T_0. The flow is slow thus we can neglect the heat transfer by conduction in flow direction. At the same time we will consider that the fluid density ρ, specific heat C_p and the heat transfer coefficient k are constant. The flow is related to a polar spatial coordinate system, the Ox axis is along the tube axis, the radial coordinate will be considered to be r and R is the radius of the tube. For the fluid velocity in the cross section we will consider the expression

$$v = v_m \cdot \frac{3\nu + 1}{\nu + 1} \cdot \left[1 - \left(\frac{r}{R}\right)^N\right], \tag{1}$$

where v_m is the mean fluid velocity, $N = (\nu + 1)/\nu$ where ν is a rheological constant of the fluid. For Newtonian fluids $\nu = 1$, for Bingham expanded fluid $\nu < 1$ and for Bingham pseudoplastic fluid $\nu > 1$.
Given these conditions the energy equation is [14]:

$$\rho C_p \mathrm{v_m} \frac{3\nu + 1}{\nu + 1}\left[1 - \left(\frac{r}{R}\right)^N\right]\frac{\partial T}{\partial x} = k\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \tag{2}$$

where K is a rheological constant of the fluid.

The aim of this article is to establish an approximate solution of equation (2), which verifies certain initial and boundary conditions.
The plan of the article is: in section two we formulate the mathematical problem, section three will contain the algorithm for determination of eigenvalues and eigenfunctions (for the Sturm-Liouville problem obtained by method of separation of variables) with Galerkin's method [15]; in the last section we will present the approximate solution of the Graetz problem and some numerical results.

The Mathematical Problem

We associate to equation (2) the initial condition

$$x = 0\,,\, T = T_0 \tag{3}$$

and boundary conditions

$$r = 0\,,\, \frac{\partial T}{\partial r} = 0\,,\, (x > 0) \tag{4}$$

$$r = R\,,\, k \cdot \frac{\partial \mathrm{T}}{\partial r} = q\,,\, (x > 0)\,. \tag{5}$$

Condition (4) specifies that at the axis of the tube has a maximum point. In (5) $q \neq 0$ is the constant wall heat flux.
It is suitable to rewrite the equation (2) and the initial and boundary conditions (3), (4), (5) in dimensionless form. With the transformation group

$$\theta = \frac{k}{q \cdot R} \cdot \left(T\text{-}T_0\right),\, \eta = \frac{r}{R}\,,\, \psi = \frac{(\nu + 1)k}{(3\nu + 1)\rho C_p R^2 v_m} x \tag{6}$$

the equation (2) and the boundary conditions (3), (4), (5) become:

$$\left(1-\eta^{N}\right)\frac{\partial\theta}{\partial\psi}=\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\theta}{\eta}\right), \quad (7)$$

$$\psi=0\,,\theta=0\,, \quad (8)$$

$$\eta=0\,,\frac{\partial\theta}{\partial\eta}=0\,,(\psi>0)\,, \quad (9)$$

$$\eta=1\,,\frac{\partial\theta}{\partial\eta}=1\,,(\psi>0)\,. \quad (10)$$

It is easy to demonstrate that a particular solution of equation (7) which verifies the conditions (9) and (10) is:

$$\theta_1=\frac{N+2}{N}\cdot\left(2\cdot\psi+\frac{1}{2}\cdot\eta^2-\frac{2}{(N+2)^2}\eta^{N+2}\right) \quad (11)$$

The change of function

$$\theta=u+\theta_1 \quad (12)$$

leads to the equation

$$\left(1-\eta^{N}\right)\frac{\partial u}{\partial\psi}=\frac{1}{r}\frac{\partial}{\partial\eta}\left(r\frac{\partial\theta}{\partial\eta}\right). \quad (13)$$

The unknown function u will satisfy the condition (9); the condition (10) becomes

$$\eta=1\,,\frac{\partial u}{\partial\eta}=0\,,(\psi>0)\,. \quad (14)$$

and the initial condition (8) is replaced by:

$$\psi=0\,,u=-\theta_1\,. \quad (15)$$

The type of equation (13) and boundary conditions (9) and (14) allow us to apply the method of separation of variables in order to determine the function u. By this method the function u is obtained under the form:

$$u(\psi,\eta)=\sum_{n=1}^{\infty}c_n\Phi_n(\eta)\exp\left(-\lambda_n^2\psi\right), \quad (16)$$

where Φ_n and λ_n are the eigenvalues and the eigenfunctions of Sturm-Liouville problem:

$$\frac{\mathrm{d}}{d\eta}\left(\eta\frac{\mathrm{d}\Phi}{\mathrm{d}\eta}\right)+\lambda^2\eta\left(1-\eta^{N}\right)\Phi=0\,, \quad (17)$$

$$\eta=0\,,\frac{\mathrm{d}\Phi}{\mathrm{d}\eta}=0\,;\eta=1\,,\frac{d\Phi}{d\eta}=0\,. \quad (18)$$

The Application of Galerkin's Method

For the determination of eigenfunctions and eigenvalues of Sturm-Liouville problem (17), (18) we will apply Galerkin's method. For this we consider the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ defined on $H_0^1(0,1)\times H_0^1(0,1)$:

$$a(u,v)=-\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta\frac{\mathrm{d}u}{\mathrm{d}\eta}\right)\cdot v\cdot \mathrm{d}\eta=\int_0^1 \eta\frac{\mathrm{d}u}{\mathrm{d}\eta}\frac{\mathrm{d}v}{\mathrm{d}\eta}\mathrm{d}\eta, \quad (19)$$
$$\mathrm{b}(u,v)=\int_0^1 \eta\cdot\left(1-\eta^N\right)\cdot u\cdot v\cdot \mathrm{d}\eta .$$

We look for the eigenpair (λ,Φ) which satisfies

$$\Phi\in H_0^1(0,1),\ \Phi\neq 0$$
$$a(\Phi,v)=\lambda^2\cdot b(\Phi,v),\ (\forall)v\in H_0^1(0,1) \quad (20)$$

(20) is called a variational formulation of (18) [16].

We look for the solution of (20) under the approximate form

$$\Phi(\eta)=\sum_{k=1}^{n} a_k\varphi_k(\eta), \quad (21)$$

where $n\in N^*$ is the approach level of function Φ and $(\varphi_k)_{k\in\mathbf{N}^*}$ is a complete system of functions in $L_2[0,1]$, functions which verify conditions

$$\varphi_k(1)=0,\ \varphi_k(\eta_0)=0,\ k\in\mathbf{N}^* . \quad (22)$$

The unknown coefficients $a_k, k=\overline{1,n}$ are determined if giving the conditions

$$a(\Phi_n,\varphi_j)=\lambda^2\cdot b(\Phi_n,\varphi_j),\ j=\overline{1,n}, \quad (23)$$

By applying these conditions we obtain the linear algebraic system in unknown a_k, $k=\overline{1,n}$:

$$\sum_{k=1}^{n}\left(\alpha_{kj}+\lambda^2\beta_{kj}\right)a_k=0,\ j=\overline{1,n}, \quad (24)$$

where

$$\alpha_{kj}=-a(\varphi_k,\varphi_j),\ j,k=\overline{1,n}, \quad (25)$$

$$\beta_{kj}=b(\varphi_k,\varphi_j),\ j,k=\overline{1,n}. \quad (26)$$

Because the system (24) must have nontrivial solutions, we obtain the equation

$$\Delta_n\equiv\left|A+\lambda^2 B\right|=0, \quad (27)$$

where A and B are the matrix $A=(\alpha_{kj})_{k,j=\overline{1,n}}$, $B=(\beta_{kj})_{k,j=\overline{1,n}}$.

The solutions of equations (27) represent the approximate values, for the n approach level, for the eigenvalues $\lambda_1^2,\ \lambda_2^2,\cdots,\lambda_n^2$.

The solution of equation (27) is difficult to be obtained under this form. Consequently, through elementary transformations of determinant Δ_n this equation takes the form [17]:

$$\left|C-\lambda^2 I_n\right|=0, \quad (28)$$

where I_n is the identity matrix of n order.

Unlike matrix A and B which are symmetric, matrix C does not have this property anymore. Therefore we must adopt an adequate method for the determination of its eigenvalues [18].

In the following we will use the complete system of functions $(\varphi_k)_{k\in\mathbf{N}^*}$ in $L_2[0,1]$:

$$\varphi_k(\eta)=J_0(\mu_k\eta), \quad (29)$$

where J_0 is the Bessel function of the first kind and zero order and $\mu_k, k \in N^*$ are the roots of equation:

$$J_1(\mu) = 0 \,. \tag{30}$$

The integrals which appear in the formulas (25), (26) are calculated with a quadrature formula that must be compatible with Galerkin's method [19]. The eigenvalues of the Sturm-Liouville problem obtained by this method are presented in the next section.
The eigenfunctions of the problem (19), (21) are the analytical form

$$\Phi_i(\eta) = \sum_{j=1}^{n} c_{ij} J_0(\mu_j \eta),\ i = \overline{1,n} \tag{31}$$

where $(c_{i1}, c_{i2}, \cdots, c_{in}), i = \overline{1,n}$ are the eigenvecteurs of the matrix $A + \lambda^2 B$.

The Approximate Solution of the Problem

The unknown function *u*, for the *n* level of approximation of Galerkin's method, is obtained from (16) and (29):

$$u(\psi,\eta) = \sum_{k=1}^{n} \left(\sum_{i=1}^{n} c_i c_{ik} \mathrm{e}^{-\lambda_i^2 \psi} \right) J_0(\mu_k \eta), \tag{32}$$

The coefficients $c_i, i = \overline{1,n}$ from (32) are determined by the use of the condition (15) and by considering that the solutions $\Phi_i, i = \overline{1,n}$ of the problem (17), (18) are orthogonal with weight $\eta(1-\eta^N)$ on $[0,1]$ [20]. Because the functions $\Phi_i, i = \overline{1,n}$ are not obtained exactly, we prefer to use the orthogonality with weight η of Bessel functions on $[0,1]$.
Thus, for the *n* level of approximation, the constants $c_i, i = \overline{1,n}$ are determined by the resolution of the linear algebraic system:

$$\sum_{i=1}^{n} c_{ik} c_i = -\frac{N+2}{N} \cdot \frac{\int_0^1 \eta \left(\frac{1}{2}\eta^2 - \frac{1}{(N+2)^2}\eta^{N+2} \right) J_0(\mu_k \eta) \mathrm{d}\eta}{\int_0^1 \eta J_0^2(\mu_k \eta) \mathrm{d}\eta}, \quad k = \overline{1,n} \tag{33}$$

The final solution of the problem is obtained now by using the relations (12), (16) and (32):

$$\theta(\psi,\eta) = \frac{N+2}{N} \cdot \left(2 \cdot \psi + \frac{1}{2} \cdot \eta^2 - \frac{2}{N+2} \eta^{N+2} \right) + \sum_{k=1}^{n} \left(\sum_{i=1}^{n} c_i c_{ik} \mathrm{e}^{-\lambda_i^2 \psi} \right) J_0(\mu_k \eta) \quad , \tag{34}$$

The eigenvalues of Sturm-Liouville problem (17), (18) are presented in table 1. The coefficients given by (25) and (26) are obtained by a numerical quadrature procedure [18]. The eigenvalues have been obtained by using the procedures BALANC, ELMHES, HQR [18]. The system (33) has been solved using a procedure based on Gauss method [18].
The variation of dimensionless temperature θ given by (34) is presented in figures 1-3. In abscisse axis there is the reduced radial distance η and in axis of ordinates there is presented the dimensionless temperature θ. The variation of dimensionless temperature θ is presented for some values of dimensionless variable ψ.

Table 1. Eigenvalues of Sturm-Liouville problem

ν									
0,35	0,5	0,6	0,7	0,75	0,8	0,9	1,0	1,1	1,2
λ_n^2									
0	0	0	0	0	0	0	0	0	0
20.942	22.427	23.247	23.966	24.293	24.602	25.169	25.679	26.141	26.560
68.283	73.102	75.800	78.175	79.260	80.283	82.168	83.863	85.395	86.788
141.810	151.783	157.384	162.323	164.581	166.712	170.637	174.170	177.366	180.271
241.465	258.414	267.946	276.356	280.201	283.832	290.522	296.544	301.993	306.947
367.227	392.974	407.462	420.251	426.098	431.622	441.799	450.961	459.253	466.792
519.085	555.452	575.924	593.997	602.262	610.070	624.458	637.411	649.135	659.795
697.032	745.840	773.322	797.588	808.685	819.170	838.490	855.885	871.630	885.948
901.064	964.136	999.656	1031.02	1045.36	1058.91	1083.89	1106.38	1126.73	1145.24
1131.17	1210.33	1254.91	1294.29	1312.29	1329.31	1360.66	1388.89	1414.45	1437.69

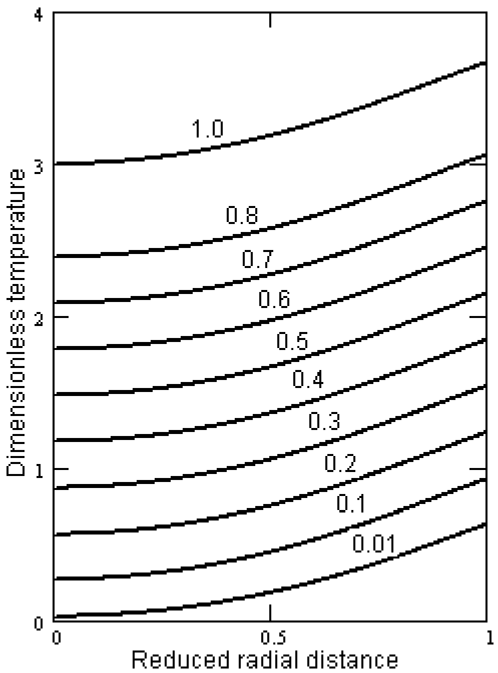

Figure 1. Dimensionless temperature profiles for ν =0,35

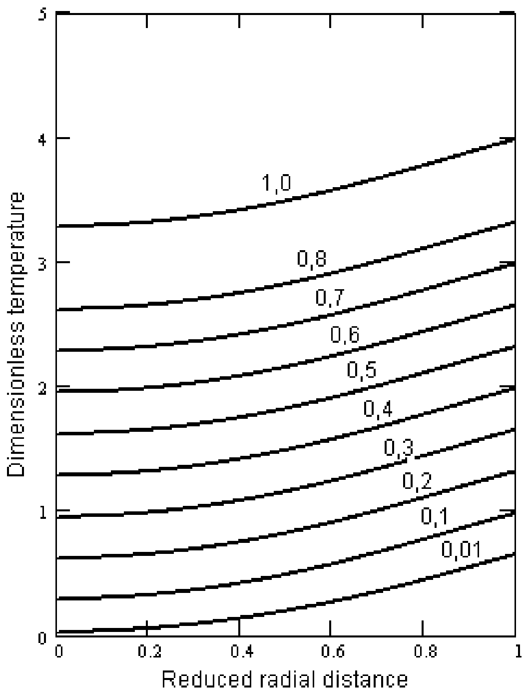

Figure 2. Dimensionless temperature profiles for ν =0,5

An important similarity criterion in the study of convective heat transfer is the Nusselt number. This number is calculated with the formula :

$$\mathrm{Nu} = -\frac{2}{<\theta> - \theta_w}, \tag{33}$$

where

$$<\theta> = \frac{\int_0^1 \eta \cdot \left(1-\eta^N\right)\theta(\eta)\,\mathrm{d}\eta}{\int_0^1 \eta \cdot \left(1-\eta^N\right)\mathrm{d}\eta} \tag{34}$$

is the bulk temperature and θ_w is the wall temperature.

In figure 4 we present the variation of Nusselt number in function of dimensionless longitudinal variable ψ and some values of parameter ν.

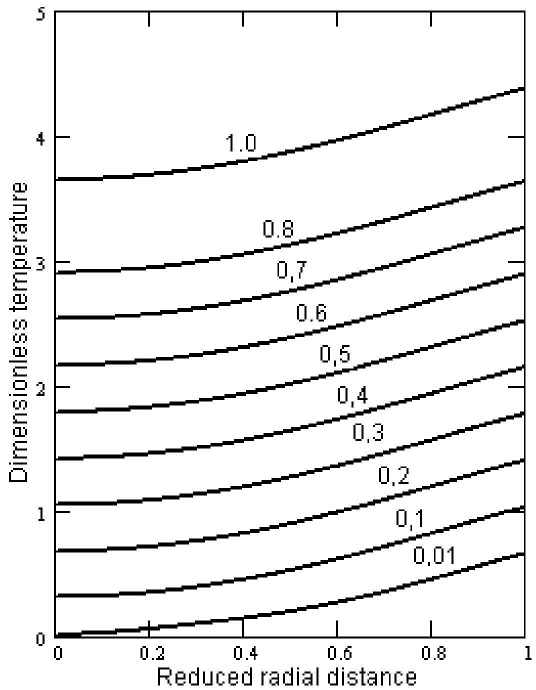

Figure 3. Dimensionless temperature profiles for ν =0,75

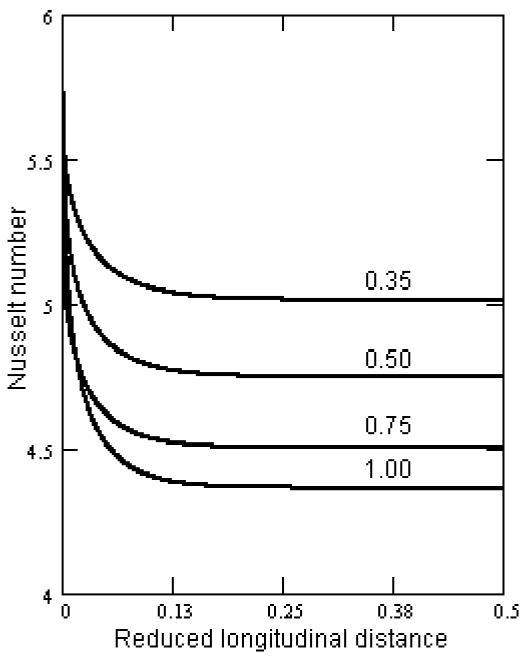

Figure 4. The variation of Nusselt number

Given the results obtained, we can deduce that for a certain value of the rheological coefficient ν, the temperature of the fluid is increased along the tube. For a given value of the dimensionless variable ψ, the temperature of the fluid is increased together with ν.

The calculations have realized for the approximation level $n = 10$ and the algorithm presents considerable stability.

As compared to [3], the paper presents the advantage of a simpler algorithm which can also be adapted to other boundary conditions (Dirichlet and Robin type conditions) by an appropriate changing of the condition (18) and of the equation (30).

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Problema Graetz pentru mişcarea fluidelor de tip putere cu condiţii la limită de tip Neumann

Rezumat

În acest articol este studiată problema Graetz in mişcarea fluidelor de tip putere printr-un tub de secţiune circulară, cu condiţii la limită de tip Neumann. Soluţia problemei este obţinută sub forma unei serii după sistemul complet de funcţii proprii al unei probleme de tip Sturm-Liouville. Valorile proprii şi funcţiile proprii ale acestei probleme Sturm-Liouville sunt obţinute cu metoda lui Galerkin.