BULETINUL Universității Petrol – Gaze din Ploiești	Vol. LX No. 1/2008	45 - 52	Seria Tehnică

Some Aspects Regarding the Elasto-Plastic Range of Beams on Elastic Foundation

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Abstract

In the paper is presented a method for calculation the length of the elasto-plastic range of rectangular beams based on elastic foundation and loaded with uniform external pressure.
The expressions of the cross sectional efforts have been established and the origin parameters have been used. In order to calculate the exact length of the elasto-plastic range, a specialised programme has been developed. The results obtained are analysed in two calculus examples.

Key words: *beam on elastic foundation, hinge, shell*

Introduction

A straight rectangular cantilever based on elastic foundation (natural sand - k_o) is considered (fig. 1).

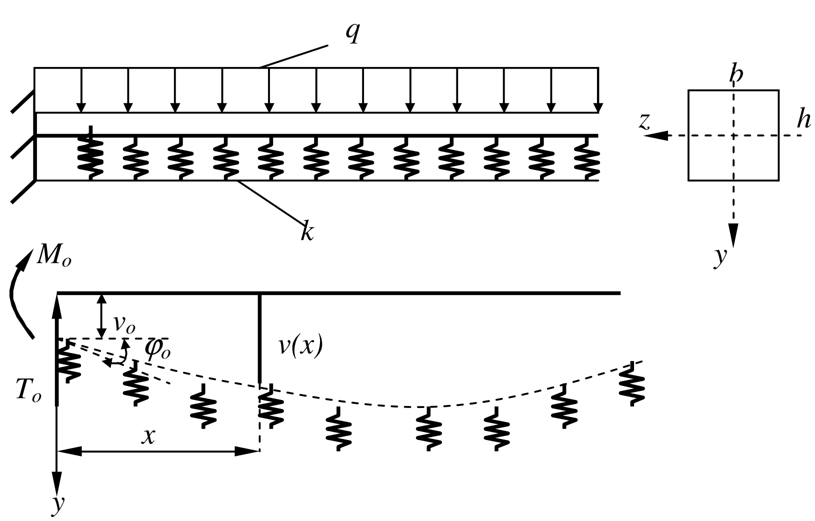

Fig. 1. The cantilever based on elastic foundation

The foundation has the rigidity k ($k = k_o \cdot b$), the beam has the origin parameters v_o, φ_o, M_o, T_o and is loaded with a uniform external pressure (fig. 1).

The differential equation that describes the deflection of the beam on elastic foundation has the following form:

$$\frac{d^4 v}{dx^4} + \frac{k}{EI} v = \frac{q}{EI} \tag{1}$$

where v represents the current deflection and the product EI the bending rigidity of the beam.
If the following notation is used :

$$\beta = \sqrt[4]{\frac{k}{4EI}} \tag{2}$$

the differential equation (1) becomes :

$$\frac{d^4 v}{dx^4} + 4\beta^4 v = \frac{q}{EI} \tag{3}$$

Solving the differential equation (3) using the methodology presented in [3] the following expression for the deflection is obtained:

$$v(x) = v_o \cdot f_1(\beta \cdot x) + \frac{\varphi_o}{\beta} \cdot f_2(\beta \cdot x) - \frac{M_o}{EI \cdot \beta^2} f_3(\beta \cdot x) - \frac{T_o}{EI \cdot \beta^3} \cdot f_4(\beta \cdot x) + \bar{v} \tag{4}$$

where $\bar{v}$ is a particular solution that corresponds to the external loads and f_1, f_2, f_3, f_4 are the Krilov functions that can be written under the forms :

$$f_1(\beta \cdot x) = ch\beta x \cdot \cos \beta x \qquad \text{(a)}$$
$$f_2(\beta \cdot x) = \frac{1}{2}(ch\beta x \cdot \sin \beta x + \cos \beta x \cdot sh\beta x) \qquad \text{(b)}$$
$$f_3(\beta \cdot x) = \frac{1}{2} sh\beta x \cdot \sin \beta x \qquad \text{(c)} \qquad (5)$$
$$f_4(\beta \cdot x) = \frac{1}{4}(ch\beta x \cdot \sin \beta x - \cos \beta x \cdot sh\beta x) \qquad \text{(d)}$$

For the case of a uniform external pressure distributed on the entire length of the beam, the particular solution has the form:

$$\bar{v} = \frac{q}{k}[1 - f_1(\beta \cdot x)] \tag{6}$$

The expressions of the slope (φ), bending moment (M) and shear force(T) can be derived from (4) and have the final forms :

$$\varphi(x)=\frac{dv}{dx}=-4\cdot\beta\cdot v_o\cdot f_4(\beta\cdot x)+\varphi_o\cdot f_1(\beta\cdot x)-\frac{M_o}{EI\cdot\beta}\cdot f_2(\beta\cdot x)-\frac{T_o}{EI\cdot\beta^2}f_3(\beta\cdot x)+\overline{\varphi} \quad (7)$$

$$M(x)=-EI\frac{d^2v}{dx^2}=4\cdot EI\cdot v_o\beta^2\cdot f_3(\beta\cdot x)+4\cdot EI\cdot\varphi_o\cdot\beta\cdot f_4(\beta\cdot x)+$$
$$+M_o\cdot f_1(\beta\cdot x)+\frac{T_o}{\beta}f_2(\beta\cdot x)+\overline{M} \quad (8)$$

$$T(x)=4\cdot EI\cdot v_o\cdot\beta^3\cdot f_2(\beta\cdot x)+4\cdot EI\cdot\varphi_o\cdot\beta^2\cdot f_3(\beta\cdot x)-4\cdot\beta\cdot M_o\cdot f_4(\beta\cdot x)+$$
$$+T_o\cdot f_1(\beta\cdot x)+\overline{T} \quad (9)$$

In the above relations, $\overline{\varphi},\overline{M},\overline{T}$ are particular solutions that can be obtained from (5) and can be written under the forms :

$$\overline{\varphi}=4\cdot\frac{\beta}{k}\cdot q\cdot f_4(\beta\cdot x) \quad \text{a)}$$
$$\overline{M}=-\frac{q}{\beta^2}f_3(\beta\cdot x) \quad \text{b)} \quad (10)$$
$$\overline{T}=-\frac{q}{\beta}f_2(\beta\cdot x) \quad \text{c)}$$

The equations (4), (7), (8) and (9) contain as unknown only the origin parameters that can be expressed from the limit conditions.

The Elasto-Plastic Range for a Cantilever

If *h'* is the height of the elastic area of the cross section of the rectangular beam (fig. 2), a

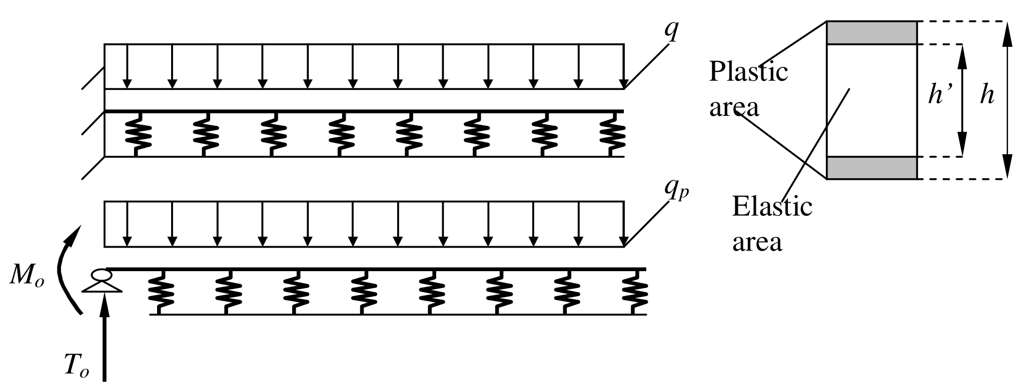

Fig. 2. The first plastic hinge in a cantilever

relation between the elasto-plastic bending moment (*M*) and the elastic bending moment (M_e) has been established in [2] :

$$\frac{M}{M_e} = \frac{3}{2} - \frac{1}{2}\left(\frac{h'}{h}\right)^2 \tag{11}$$

The abscissa where $h' = 0$ corresponds to the plastification of the entire section and the abscissa where $h' = h$ to the case that the entire section is elastic.
In order to calculate the length of the elasto-plastic range it is necessary to be expressed the elasto-plastic bending moment, using the hypothesis that in the starting point (where the beam is embedded and the moment is maximum) appears the first plastic hinge. This hypothesis has to be confirmed by the relation (10) too.

For a cantilever (that is embedded at one end and free at the other one) the limit conditions have the forms :

$$x = 0 \Rightarrow \begin{cases} v = v_o = 0 & \text{a)} \\ \varphi = \varphi_o = 0 & \text{b)} \end{cases} \tag{12}$$

$$x = l \Rightarrow \begin{cases} M(l) = 0 & \text{a)} \\ T(l) = 0 & \text{b)} \end{cases} \tag{13}$$

Taking into consideration the above limit conditions and the relations (8) and (9), the following system of equations(in unknown M_o and T_o) is obtained :

$$\begin{aligned} & M_o \cdot f_1(\beta \cdot l) + \frac{T_o}{\beta} f_2(\beta \cdot l) = \frac{q}{\beta^2} f_3(\beta \cdot l) && \text{(a)} \\ & -4 \cdot M_o \cdot \beta \cdot f_4(\beta \cdot l) + T_o \cdot f_1(\beta \cdot l) = \frac{q}{\beta} f_2(\beta \cdot l) && \text{(b)} \end{aligned} \tag{14}$$

Solving the above system of linear equations, the following expressions for M_o and T_o are obtained :

$$\begin{aligned} & M_o = \frac{f_1(\beta \cdot l) \cdot f_3(\beta \cdot l) - [f_2(\beta \cdot l)]^2}{\beta^2 \{[f_1(\beta \cdot l)]^2 + 4 \cdot f_2(\beta \cdot l) \cdot f_4(\beta \cdot l)\}} \cdot q = M_o^A \cdot q && \text{a)} \\ & T_o = \frac{f_1(\beta \cdot l) \cdot f_2(\beta \cdot l) + 4 \cdot f_3(\beta \cdot l) \cdot f_4(\beta \cdot l)}{\beta \{[f_1(\beta \cdot l)]^2 + 4 \cdot f_2(\beta \cdot l) \cdot f_4(\beta \cdot l)\}} \cdot q = T_o^A \cdot q && \text{b)} \end{aligned} \tag{15}$$

Replacing (15a) into (11) is obtained an equality that allows the calculation of the length of the elasto-plastic range:

$$\frac{M_{op} \cdot f_1(\beta \cdot x) + \frac{T_{op}}{\beta} \cdot f_2(\beta \cdot x) - \frac{q_p}{\beta^2} \cdot f_3(\beta \cdot x)}{M_{oe}} = \frac{3}{2} - \frac{1}{2}\left(\frac{h'}{h}\right)^2 \tag{16}$$

In (16) M_{op} and T_{op} are the reactions in the starting point produced by the pressure q_p that produces a plastic hinge in the starting point and M_{oe} is the elastic bending moment from the same starting point.

Because the M_{op} and T_{op} are linearly expressed by q (15), it can be noticed that, in equality (15) can be emphasised the ratio between the plastic and elastic pressure (q_p/q_e) that, for the rectangular cross sections has the value 1.5 [2].

If the numerical coefficients that multiply q in (15) are respectively M_o^A and T_o^A, the equality (16) becomes:

$$\frac{M_o^A \cdot f_1(\beta \cdot x) + \frac{T_o^A}{\beta} \cdot f_2(\beta \cdot x) - \frac{1}{\beta^2} \cdot f_3(\beta \cdot x)}{M_o^A} = 1 - \frac{1}{3}\left(\frac{h'}{h}\right)^2 \quad (17)$$

If $h' = 0$ the elastic area of the cross section does not exists and the cross section is totally plastified. This happens for $x = 0$ and the hypothesis that the first hinge appears in the embedded point (starting point) is confirmed.

The elasto-plastic range ends in the section where $h' = h$ (the entire cross section is elastic). For this situation, from (16) can be developed the equation :

$$f(x) = f_1(\beta \cdot x) + \frac{T_o^A}{\beta \cdot M_o^A} \cdot f_2(\beta \cdot x) - \frac{1}{\beta^2 \cdot M_o^A} \cdot f_3(\beta \cdot x) - \frac{2}{3} = 0 \quad (18)$$

The first positive root of the above equation represents the length of the elasto-plastic range of the beam.

The Elasto-Plastic Range for a Cantilever Embedded at One End and Simply Supported at the Other One

For such a beam that is based on a elastic foundation (fig. 3) the limit conditions are:

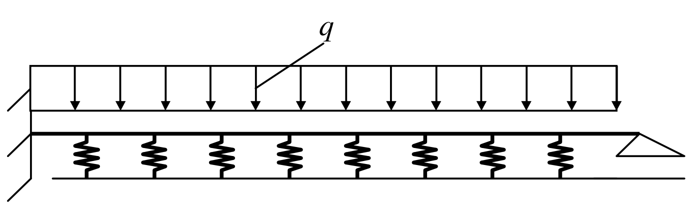

Fig.3. Beam on elastic foundation embedded at one end and simply supported at the other one

$$x = 0 \Rightarrow \begin{cases} v = v_o = 0 & \text{a)} \\ \varphi = \varphi_o = 0 & \text{b)} \end{cases} \quad (19)$$

$$x = l \Rightarrow \begin{cases} v(l) = 0 & \text{a)} \\ M(l) = 0 & \text{b)} \end{cases} \quad (20)$$

Replacing the relations (4) and (8) into (20), the following system of equations is obtained :

$$M_o \cdot f_3(\beta \cdot l) + \frac{T_o}{\beta} f_4(\beta \cdot l) = \frac{q}{4\beta^2}\left[1 - f_1(\beta \cdot l)\right] \qquad \text{(a)}$$

$$M_o \cdot f_1(\beta \cdot l) + \frac{T_o}{\beta} \cdot f_2(\beta \cdot l) = \frac{q}{\beta^2} \cdot f_3(\beta \cdot l) \qquad \text{(b)} \qquad (21)$$

Solving the above system the constants M_o and T_o are :

$$M_o = \frac{f_2(\beta \cdot l) - f_1(\beta \cdot l) \cdot f_2(\beta \cdot l) - 4 \cdot f_3(\beta \cdot l) \cdot f_4(\beta \cdot l)}{4 \cdot \beta^2 \cdot \left[f_2(\beta \cdot l) \cdot f_3(\beta \cdot l) - f_1(\beta \cdot l) \cdot f_4(\beta \cdot l)\right]} \cdot q = M_o^B \cdot q \qquad \text{a)}$$

$$T_o = \frac{4\left[f_3(\beta \cdot l)\right]^2 + \left[f_1(\beta \cdot l)\right]^2 - f_1(\beta \cdot l)}{4 \cdot \beta \cdot \left[f_2(\beta \cdot l) \cdot f_3(\beta \cdot l) - f_1(\beta \cdot l) \cdot f_4(\beta \cdot l)\right]} \cdot q = T_o^B \cdot q \qquad \text{b)} \qquad (22)$$

The relations (22) have to be used in order to establish if the maximum bending moment is obtained in the starting point, that is associated with the first plastic hinge.

If the bending moment reaches his maximum in the embedded point (starting point) the length of the elasto-plastic range is the first positive root of the (18) equation.

Calculus Examples

It is considered a rectangular beam (200 x 200 *mm*) made of wood ($E = 10^4$ *N/mm*^{2}) based on an elastic foundation (natural sand: $k_o = 0.05$ *N/mm*^{3}). The beam has the length $l = 4$ *m*, it is loaded by an uniform external pressure q (*N/mm*), and is analysed in two cases: when it is embedded at one end and free at the other one (fig. 4a) and when it is embedded at one end and simply supported at the other one (fig. 4b).

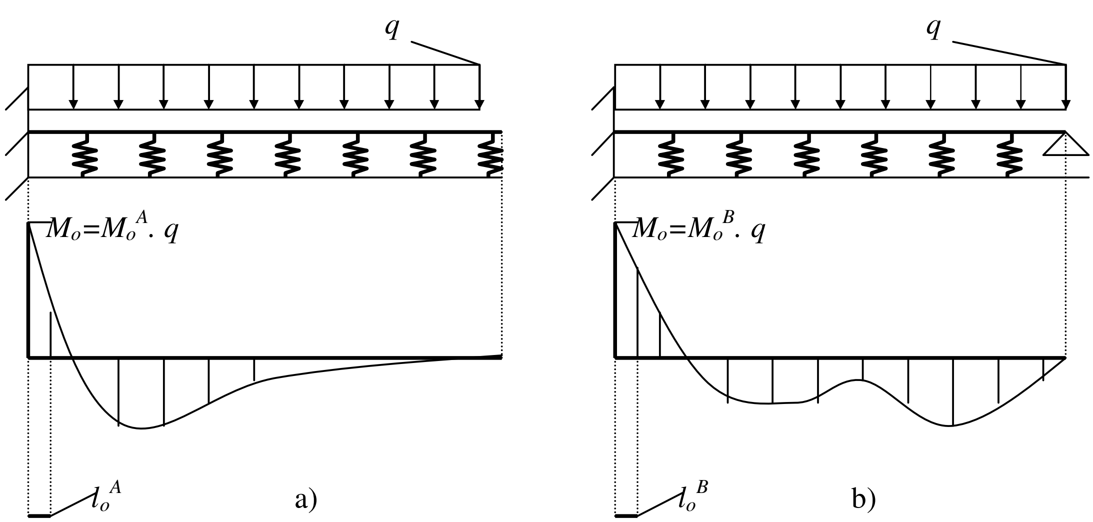

Fig. 4. Bending diagrams for calculus examples

The rigidity of elastic foundation connected with the beam is obtained multiplying the coefficient k_o with the depth of the cross section of the beam, b:

$$k = k_o \cdot b = 0.05 \cdot 200 = 10\ N/mm^2$$

The bending rigidity of the beam is:

$$EI = E \cdot \frac{b \cdot b^3}{12} = 10^4 \cdot \frac{200^4}{12} = 1.33 \cdot 10^{12}\ \ N \cdot mm^2$$

The β coefficient is calculated with the relation (2):

$$\beta = \sqrt[4]{\frac{k}{4 \cdot EI}} = \sqrt[4]{\frac{10}{4 \cdot 1.33 \cdot 10^{12}}} = 0{,}0011701736\ \frac{1}{mm}$$

In order to calculate the variation of the bending moments and the lengths of the elasto-plastic ranges, a specialised programme has been developed. The graphics of the bending moments are represented in figure 4 for the two analysed cases.

It can be noticed that, in both cases the bending moments reach the maximum values in the starting point (in the embedded point), that means that the first plastic hinge appears in this initial point.

Using the function (18), the following lengths for the elasto-plastic ranges have been obtained :

- for the case presented in figure 4a : $l_o = 156.8\ mm$;
- for the case presented in the figure 4b : $l_o = 160\ mm$.

It can be noticed that, in the both cases, the lengths of the elasto-plastic ranges are nearly the same and represent a small percent of the entire length of the beam.

This phenomenon can be explained by the presence of the elastic foundation that makes the bending of the beam to be smaller.

Conclusions

In the paper, it is presented a method that allows the exact calculation of the length of the elasto-plastic range for a rectangular beam based on an elastic foundation.

This range is important especially for the calculus of the deformed shape of the beam, where it is necessary to know exactly the elastic and the elasto-plastic intervals.

The results obtained are applied in two calculus examples: when the cantilever is embedded at the starting point and is free at the other one and when the cantilever is embedded at the starting point and simply supported at the other one. In both cases the beam is also supported on an elastic foundation.

In both cases presented above the lengths of the elasto-plastic intervals are nearly the same and represent a small percent from the length of the beam. This result can be explained by the presence of the elastic foundation that allows the beam to be plastified only in a small interval located in the neighbourhood of the embedded point (starting point in these examples).

The results obtained can be also used in the shell theory (for cylinders) because the bending of a cylindrical shell can be described by the same differential equation with the beams on elastic foundation.

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Aspecte privind determinarea zonelor elasto-plastice la grinzi pe mediu elastic

Rezumat

În lucrare se prezintă o metodologie de determinare a lungimii zonelor elasto-plastice la barele drepte cu secţiune dreptunghiulară, aşezate pe o fundaţie elastică şi solicitate de o sarcină uniform distribuită pe toată lungimea. Sunt utilizate expresiile eforturilor secţionale exprimate în funcţie de parametrii în origine şi este realizat un program de calcul care permite calculul exact pe care bara se află în domeniul elasto-plastic. Lungimea acestui interval este importantă în calculul deplasărilor, care se determină diferit în funcţie de stadiul elastic sau elasto-plastic.

Rezultatele obţinute sunt analizate pe exemple concrete, cu diverse rezemări la capete.

Se sugerează, de asemenea, că rezultatele obţinute pot fi aplicate şi pentru cilindrii subţiri presurizaţi interior.