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On Viscous Dissipation in the Incompressible Fluid Flow Between Two Parallels Plates **with Dirichlet Boundary Conditions**

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Abstract

This paper considers the problem of viscous dissipation in the laminar incompressible fluid flow between two parallels plates. The method proposed to determine the temperature of the fluid makes use of the separation of variables. Thus the solution of the problem is obtained by series expansion about the complete eigenfunctions system of a Sturm-Liouville problem. Eigennfunction and eigenvalues of this Sturm-Liouville problem are obtained by Galerkin's method.

Key words: *dissipation, power law fluid, eigenfunction, Galerkin's method.*

Introduction

The problem of viscous dissipation in the fluid flow has many practical applications. An example is oil products transport through ducts; another is the polymers processing.

Now we will consider the incompressible laminar fluid flow between two infinite parallel plates. The plates are maintained at a constant temperature, T_0, and the fluid flows through the plates with the same temperature. The flow is slow, thus we can neglect the heat transfer by conduction in the flow direction. At the same time we will consider that the fluid density, ρ, specific heat, C_p, and the heat transfer coefficient, k, are constants. The flow is related to a cartesian coordinate system, the Ox axis is directed to the flow direction, the Oy axis is normal to the plates and the distance between plates is $2h$.

For the fluid velocity in the cross section we will consider the expression:

$$v = v_0\left[1-\left(\frac{y}{h}\right)^N\right], \tag{1}$$

where v_0 is the maximal fluid velocity, $N=(n+1)/n$ where n is a rheological constant of the fluid. For Newtonian fluids $n=1$, for Bingham expanded fluid $n<1$, and for Bingham pseudoplastic fluid $n>1$.

Given these conditions, the energy equation is [1], [2]:

$$\rho C_p v \frac{\partial T}{\partial x} = k \frac{\partial^2 T}{\partial y^2} + K\left(-\frac{\partial v}{\partial y}\right)^{n+1} \tag{2}$$

where K is a rheological constant of the fluid.

The aim of this article is to establish an approximate solution of equation (1), which verifies certain initial and boundary conditions.

The plan of the article is: in section two we formulate the mathematical problem, section three will contain the algorithm for determination of eigenvalues and eigenfunctions (for the Sturm-Liouville problem obtained using the method of separation of variables) with Galerkin's method [3]; in the last section, we will present the approximate solution of the problem and some numerical results.

The Mathematical Problem

We associate to equation (1) the initial condition

$$x = 0\,,\, T = T_0 \tag{3}$$

and boundary conditions

$$y = 0\,,\, \frac{\partial T}{\partial y} = 0\,,\, (x > 0) \tag{4}$$

$$y = h\,,\, \mathrm{T} = \mathrm{T}_0\,,\, (x > 0) \tag{5}$$

Condition (4) specifies that at the middle of the distance between plates the temperature has a maximum point.

It is suitable to rewrite the equation (2) and the initial and boundary conditions (3), (4), (5) in dimensionless form. With the transformation group:

$$\theta = \frac{T\text{-}T_0}{T_0}\,,\, \eta = \frac{y}{h}\,,\, \psi = \frac{kx}{\rho C_p H^2 v_0} \tag{6}$$

the equation (2) and the boundary conditions (3), (4), (5) becomes:

$$\left(1-\eta^N\right)\frac{\partial\theta}{\partial\psi} = \frac{\partial^2\theta}{\partial\eta^2} + N_{Br}\eta^N\,, \tag{7}$$

$$\psi = 0\,,\, \theta = 0\,, \tag{8}$$

$$\eta = 0\,,\, \frac{\partial\theta}{\partial\eta} = 0\,,\, (\psi > 0)\,, \tag{9}$$

$$\eta = 1\,,\, \theta = 0\,,\, (\psi > 0)\,. \tag{10}$$

In equation (7) the coefficient N_{Br} is the Brinkman number [2].

It is easy to find that a particular solution of equation (7) which verifies condition (10) is:

$$\theta_1 = \frac{N_{Br}}{(N+1)(N+2)}\left(1-\eta^{N+2}\right) \tag{11}$$

The change of function

$$\theta = u + \theta_1 \tag{12}$$

leads to the equation

$$\left(1-\eta^N\right)\frac{\partial u}{\partial \psi}=\frac{\partial^2 u}{\partial \eta^2}. \tag{13}$$

The unknown function u will satisfy the conditions (9) and (10) and the initial condition (8) is replaced by:

$$\psi = 0\,,\ u = -\theta_1\,. \tag{14}$$

The type of equation (13) and boundary conditions (9) and (10) allow us to apply the method of separation of variables in order to determine function u. By this method function u is obtained under the form:

$$u(\psi,\eta)=\sum_{n=1}^{\infty} c_n \Phi_n(\eta)\exp\left(-\lambda_n^2\psi\right)\,, \tag{15}$$

where Φ_n and λ_n are the eigenvalues and the eigenfunctions of Sturm-Liouville problem:

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}\eta^2}+\lambda^2\left(1-\eta^N\right)\Phi=0\,, \tag{16}$$

$$\eta=0\,,\ \frac{\mathrm{d}\Phi}{\mathrm{d}\eta}=0\,;\ \eta=1\,,\ \Phi=0\,. \tag{17}$$

The Application of Galerkin's Method

For the determination of eigenfunctions and eigenvalues of Sturm-Liouville problem (16), (17) we will apply the Galerkin's method. For this we consider the operator:

$$U : D(U)\subset L_2[0,1]\to L_2[0,1],$$
$$D(U)=\left\{\Phi\in C^2[0,1]\,,\ \frac{\mathrm{d}\Phi}{\mathrm{d}\eta}(0)=0\,,\ \Phi(1)=0\right\}, \tag{18}$$
$$U(\Phi)=\frac{\mathrm{d}^2\Phi}{\mathrm{d}\eta^2}+\lambda^2\left(1-\eta^N\right)\Phi\,.$$

We look at the solution of Sturm-Liouville problem (16), (17) under the approximate form

$$\Phi(\eta)=\sum_{k=1}^{n} a_k \varphi_k(\eta)\,, \tag{19}$$

where $n\in\mathbf{N}^*$ is the approach level of function Φ, and $(\varphi_k)_{k\in\mathbf{N}^*}$ is a complete system of functions in $L_2[0,1]$, functions which verify the conditions [4]:

$$\frac{\mathrm{d}\varphi_k}{\mathrm{d}\bar{r}}(0)=0\,,\ \varphi_k(1)=0\,,\ k\in\mathbf{N}^*\,. \tag{20}$$

The unknown coefficients a_k, $k=\overline{1,n}$ are determined, given the conditions

$$< U(\Phi),\varphi_j > = 0\,,\ j=\overline{1,n}\,, \tag{21}$$

the scalar product being considered in the space of square integrable function $L_2[0,1]$.

By applying these conditions, we obtain the linear algebraic system in unknown a_k, $k=\overline{1,n}$:

$$\sum_{k=1}^{n}\left(\alpha_{kj}+\lambda^2\beta_{kj}\right)a_k=0\,,\ j=\overline{1,n}\,, \tag{22}$$

where

$$\alpha_{kj}=\int_0^1\frac{\mathrm{d}^2\varphi_k}{\mathrm{d}\eta^2}\varphi_j\mathrm{d}\eta\,,\ j,k=\overline{1,n}\,, \tag{23}$$

$$\beta_{kj}=\int_0^1\left(1-\eta^N\right)\varphi_k\varphi_j\mathrm{d}\eta\,,\ j,k=\overline{1,n}\,. \tag{24}$$

Because the system (22) must have nontrivial solutions, we obtain the equation:

$$\Delta_n\equiv\left|A+\lambda^2B\right|=0\,, \tag{25}$$

where A and B are the matrix $A=\left(\alpha_{kj}\right)_{k,j=\overline{1,n}}$, $B=\left(\beta_{kj}\right)_{k,j=\overline{1,n}}$. The solutions of equations (25) represent the approximate values, for the n approach level, for the eigenvalues $\lambda_1^2,\ \lambda_2^2,\Lambda\ ,\lambda_n^2$.

The solution of equation (1) is difficult to be obtained under this form. Consequently, through elementary transformations of determinant Δ_n, this equation takes the form [5]:

$$\left|C-\lambda^2I_n\right|=0\,, \tag{26}$$

where I_n is the identity matrix of n order.

Unlike matrix A and B which are symmetric, matrix C does not have this property anymore. Therefore we must adopt an adequate method for the determination of its eigenvalues [6].

In the following we will use the complete system of functions $\left(\varphi_k\right)_{k\in\mathbf{N}^*}$ in $L_2[0,1]$:

$$\varphi_k(\eta)=J_0(\mu_k\eta)\,, \tag{27}$$

where J_0 is the Bessel function of the first kind and zero order and $\mu_k,\ k\in\mathbf{N}^*$ are the roots of equation:

$$J_0(\mu)=0\,. \tag{28}$$

The integrals which appear in the formulas (23), (24) are calculated with a quadrature formula that must be compatible with Galerkin's method [7]. The eigenvalues of the Sturm-Liouville problem obtained by this method are presented in the next section.

The eigenfunctions of the problem (18), (19) are the analytical form

$$\Phi_i(\eta)=\sum_{j=1}^{n}c_{ij}J_0\left(\mu_j\eta\right),\ i=\overline{1,n} \tag{29}$$

where $\left(c_{i1},c_{i2},\Lambda\ ,c_{in}\right),\ i=\overline{1,n}$ are the eigenvecteurs of the matrix $A+\lambda^2B$.

The Approximate Solution of the Problem

The unknown function u, for the n level of approximation of Galerkin's method, is obtained from (15) and (27):

$$u(\psi,\eta)=\sum_{k=1}^{n}\left(\sum_{i=1}^{n}c_i c_{ik}\mathrm{e}^{-\lambda_i^2\psi}\right)J_0(\mu_k\eta), \tag{30}$$

The coefficients c_i , $i=\overline{1,n}$ from (30) are determined by the use of the condition (14) and by considering that the solutions $\Phi_i, i=\overline{1,n}$, of the problem (16), (17) are orthogonal with weight $1-\eta^N$ by [0,1] [4]. Because the functions $\Phi_i, i=\overline{1,n}$, are not obtained exactly, we prefer to use the orthogonality with weight η of Bessel functions on [0,1]. Thus, for the n level of approximation, the constants c_i , $i=\overline{1,n}$ are determined by the resolution of the linear algebraic system:

$$\sum_{i=1}^{n}c_{ik}c_i=-\frac{N_{Br}}{(N+1)(N+2)}\frac{\int_0^1\left(1-\eta^{N+2}\right)\eta J_0(\mu_k\eta)\mathrm{d}\eta}{\int_0^1\eta J_0^2(\mu_k\eta)\mathrm{d}\eta},\ k=\overline{1,n}, \tag{31}$$

The final solution of the problem is obtained now by using the relations (12), (15) and (30):

$$\theta(\psi,\eta)=\frac{N_{Br}}{(N+1)(N+2)}\left(1-\eta^{N+2}\right)+\sum_{k=1}^{n}\left(\sum_{i=1}^{n}c_i c_{ik}\mathrm{e}^{-\lambda_i^2\psi}\right)J_0(\mu_k\eta), \tag{32}$$

As an example, we will consider a fluid with unit Brinkman number. The eigenvalues of Sturm-Liouville problem (16), (17) are presented in table 1. The coefficients given by (23) and (24) are obtained by a numerical quadrature procedure [6]. The eigenvalues have been obtained by using the procedures BALANC, ELMHES, HQR [6]. System (31) has been solved using a procedure based on Gauss method [6].

Table 1. Eigenvalues of Sturm-Liouville problem

n									
0,35	0,5	0,6	0,7	0,75	0,8	0,9	1,0	1,1	1,2
λ_n^2									
2,578	2,646	2,688	2,727	2,746	2,764	2,797	2,827	2,855	2,881
26,368	28,132	29,133	30,019	30,425	30,807	31,512	32,147	32,720	33,242
76,320	81,567	84,535	87,161	88,362	89,498	91,590	93,474	95,179	96,730
152,343	162,900	168,861	174,131	176,543	178,822	183,022	186,805	190,229	193,343
254,430	272,117	282,097	290,917	294,953	298,767	305,796	312,127	317,858	323,070
382,605	409,243	424,267	437,544	443,618	449,358	459,938	469,467	478,093	485,938
536,843	574,258	595,355	613,996	622,525	630,584	645,437	658,815	670,925	681,940
717,133	767,145	795,342	820,254	831,652	842,421	862,271	880,148	896,332	911,051
923,497	987,926	1024,24	1056,33	1071,01	1084,88	1110,45	1133,48	1154,32	1173,28
1155,92	1236,59	1282,06	1322,23	1340,61	1357,98	1389,98	1418,81	1444,90	1468,64

The variation of the dimensionless temperature θ given by (32) is presented in figures 1-5. In the abscisse axis, there is the reduced transverse distance η and in the axis of ordinates it is presented the dimensionless temperature θ. The variation of dimensionless temperature θ is presented for some values of dimensionless variable ψ.

An important similarity criterion in the study of convective heat transfer is the Nusselt number. This number is calculated with the formula [8]:

$$\mathrm{Nu} = -\frac{2\cdot \left.\frac{\partial\theta}{\partial\eta}\right|_{\eta=1}}{<\theta> - \theta_w}, \tag{33}$$

where

$$<\theta> = \frac{\int_0^1 \left(1-\eta^N\right)\theta(\eta)\,d\eta}{\int_0^1 \left(1-\eta^N\right)d\eta} \tag{34}$$

is the bulk temperature and θ_w is the wall temperature.

In figure 6, we present the variation of Nusselt number in function of dimensionless longitudinal variable ψ and some values of parameter *n*.

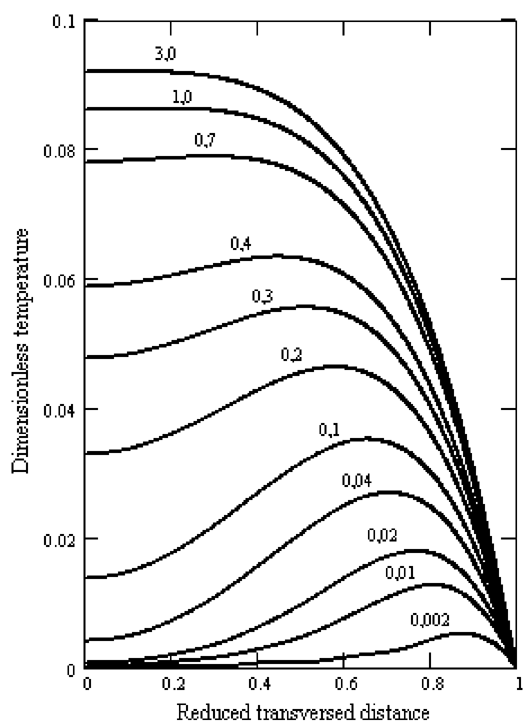

Fig. 1. Dimensionless temperature profiles for constant walls temperature, n=1,2, N_{Br}=1

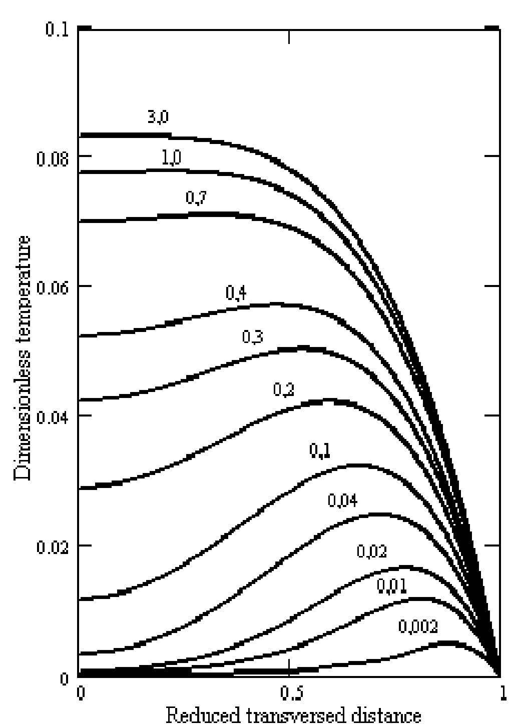

Fig. 2. Dimensionless temperature profiles for constant walls temperature, n=1,0 (Newtonian fluid), N_{Br}=1

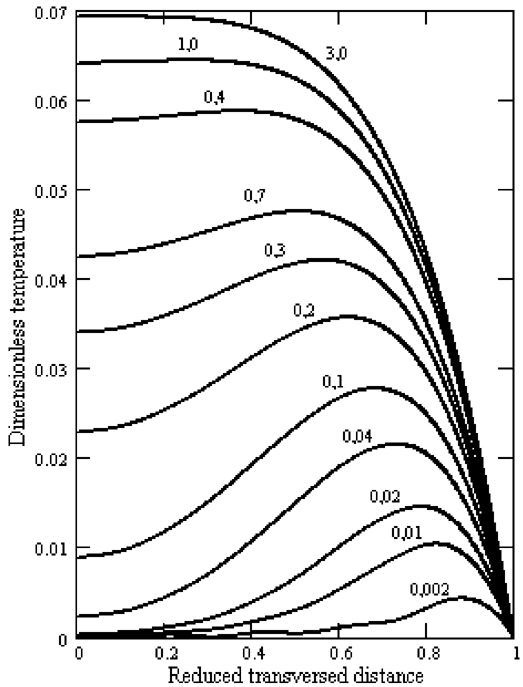

Fig. 3. Dimensionless temperature profiles for constant walls temperature, n=0,75, N_{Br}**=1**

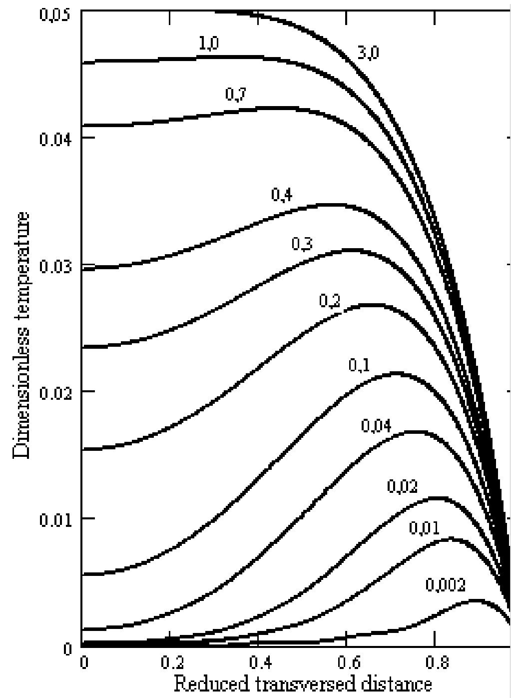

Fig. 4. Dimensionless temperature profiles for constant walls temperature, n=0,5, N_{Br}=1

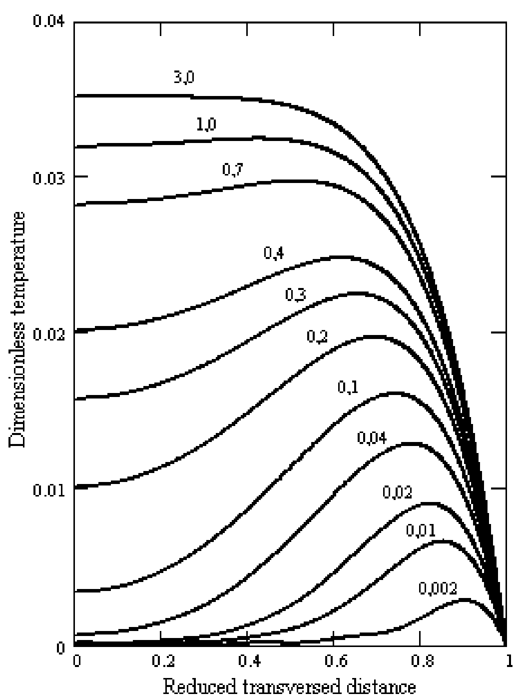

Fig. 5. Dimensionless temperature profiles for constant walls temperature, n=0,35, N_{Br}=1

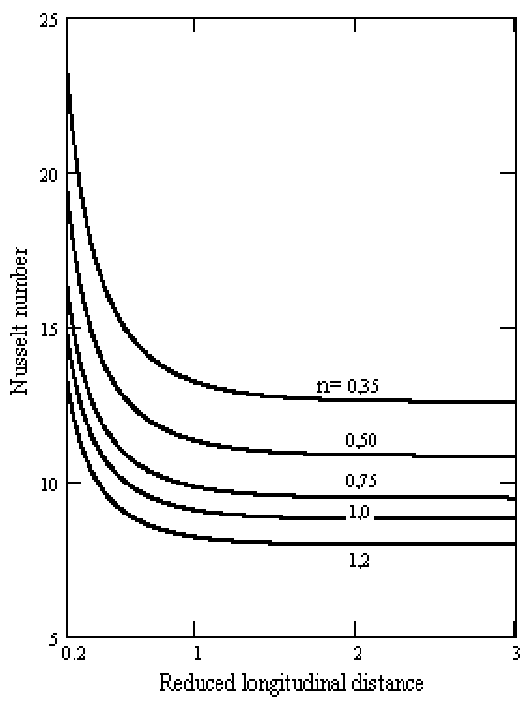

Fig. 6. Variation of Nusselt number with reduced longitudinal distance, N_{Br}=1

Given the results obtained, we can deduce that for a certain value of the rheological coefficient *n,* the temperature of the fluid is increased along the plates. For a given value of the dimensionless variable ψ, the temperature of the fluid is increased together with *n*. As opposed to it, the Nusselt number decreases white *n* increases. The obtained results properly fit the results in [2].

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Asupra disipaţiei vâscoase în mişcarea fluidelor incompresibile printre două plăci plane paralele cu condiţii la limită Dirichlet

Rezumat

În acest articol este studiată problema disipaţiei vâscoase în mişcarea laminară incompresibilă a unui fluid vâscos printre două plăci plane paralele. Se utilizează pentru determinarea temperaturii fluidului metoda separării variabilelor. Soluţia problemei se obţine astfel sub forma unei serii după sistemul complet de funcţii proprii unei probleme de tip Sturm-Liouville. Funcţiile şi valorile proprii ale acestei probleme Sturm-Liouville sunt obţinute cu ajutorul metodei lui Galerkin.