BULETINUL Universităţii Petrol – Gaze din Ploieşti	Vol. LVIII No. 3/2006	19 - 26	Seria Tehnică

On Viscous Dissipation in Power-Law Fluid Flow through a Circular Cross Section Tube

Tudor Boacă

Universitatea Petrol – Gaze din Ploieşti, Bd. Bucureşti 39, Ploieşti
e-mail: tboaca@upg-ploiesti.ro

Abstract

This paper considers the problem of viscous dissipation in the flow of power-law fluid through a tube of circular cross section, with Neumann boundary conditions. The solution of the problem is obtained by a series expansion about the complete eigenfunctions system of a Sturm-Liouville problem. Eigenfunctions and eigenvalues of this Sturm-Liouville problem are obtained by Galerkin's method.

Key words: *dissipation, power law fluid, eigenfunction, Galerkin's method*

Introduction

The problem of viscous dissipation in the fluid flow through a tube of circular cross section has many practical applications. An example is oil product transport through ducts; another is the polymer processing [1].

The problem has constituted the object of many researches. Various approximate methods have been proposed for the determination of its solution. Recently, Valko [2] has obtained an approximate solution by means of a combined method which uses the Laplace transform and Galerkin method. Other approaches of the problem have been given in [3], [4], [5].

In [6] we have obtained an approximate solution of this problem in the case of Dirichlet boundary conditions.

In this paper, we will consider the flow of power-law fluid through a tube of circular cross section with Neumann boundary conditions (adiabatic wall). At the entrance of the tube the temperature of the fluid is T_0. The flow is slow thus we can neglect the heat transfer by conduction in flow direction. At the same time, we will consider that the fluid density ρ, specific heat C_p and the heat transfer coefficient k are constant. The flow is related to a polar spatial coordinate system, the Ox axis is along the tube axis, the radial coordinate will be considered to be *r,* and R is the radius of the tube. For the fluid velocity in the cross section we will consider the expression

$$v = v_m \cdot \frac{3\nu+1}{\nu+1} \cdot \left[1 - \left(\frac{r}{R} \right)^N \right], \tag{1}$$

where v_m is the mean fluid velocity, $N=(\nu+1)/\nu$, where ν is a rheological constant of the fluid. For Newtonian fluids $\nu = 1$, for Bingham expanded fluid $\nu < 1$, and for Bingham pseudoplastic fluid $\nu > 1$.

Given these conditions the energy equation is [7], [2]:

$$\rho C_p \mathrm{v}_{\mathrm{m}} \frac{3\nu+1}{\nu+1}\left[1-\left(\frac{r}{R}\right)^N\right]\frac{\partial T}{\partial x} = k\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+K\left|\frac{\partial v}{\partial r}\right|^{\nu+1}, \tag{2}$$

where K is a rheological constant of the fluid.

The aim of this paper is to establish an approximate solution of equation (2) which verifies certain initial and boundary conditions.

The plan of the article is: in section two, we formulate the mathematical problem; section three will contain the algorithm for the determination of eigenvalues and eigenfunctions (for the Sturm-Liouville problem obtained by the method of separation of variables) with Galerkin's method [8]; in the last section, we will present the approximate solution of the problem and some numerical results.

The Mathematical Problem

We associate to equation (2) the initial condition

$$x=0\,,\, T=T_0 \tag{3}$$

and boundary conditions

$$r=0\,,\,\frac{\partial T}{\partial r}=0\,,\,(x>0) \tag{4}$$

$$y=R\,,\,\frac{\partial \mathrm{T}}{\partial r}=0\,,\,(x>0)\,. \tag{5}$$

Condition (4) specifies that at the axis of the tube has a maximum point.

It is suitable to rewrite the equation (2) and the initial and boundary conditions (3), (4), (5) in dimensionless form. With the transformation group

$$\theta=\frac{T\text{-}T_0}{T_0}\,,\,\eta=\frac{r}{R}\,,\,\psi=\frac{(\nu+1)k}{(3\nu+1)\rho C_p R^2 v_m}x \tag{6}$$

the equation (2) and the boundary conditions (3), (4), (5) become:

$$\left(1-\eta^N\right)\frac{\partial\theta}{\partial\psi}=\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\theta}{\eta}\right)+N_{Br}\eta^N\,, \tag{7}$$

$$\psi=0\,,\,\theta=0\,, \tag{8}$$

$$\eta=0\,,\,\frac{\partial\theta}{\partial\eta}=0\,,\,(\psi>0)\,, \tag{9}$$

$$\eta=1\,,\,\frac{\partial\theta}{\partial\eta}=0\,,\,(\psi>0)\,. \tag{10}$$

In equation (7), the coefficient N_{Br} is the Brinkman number [1], [2].

It is easy to demonstrate that a particular solution of equation (7) which verifies the conditions (9) and (10) is:

$$\theta_1 = \frac{N_{Br}}{2N}\left(4\psi + \eta^2 - \frac{2}{N+2}\eta^{N+2}\right) \tag{11}$$

The change of function

$$\theta = u + \theta_1 \tag{12}$$

leads to the equation

$$\left(1-\eta^N\right)\frac{\partial u}{\partial \psi} = \frac{1}{r}\frac{\partial}{\partial \eta}\left(r\frac{\partial \theta}{\partial \eta}\right). \tag{13}$$

The unknown function u will satisfy the conditions (9) and (10), and the initial condition (8) is replaced by:

$$\psi = 0\,,\; u = -\theta_1\,. \tag{14}$$

The type of equation (13) and boundary conditions (9) and (10) allow us to apply the method of separation of variables in order to determine the function u. By this method, the function u is obtained under the form:

$$u(\psi,\eta) = \sum_{n=1}^{\infty} c_n \Phi_n(\eta)\exp\left(-\lambda_n^2\psi\right), \tag{15}$$

where Φ_n and λ_n are the eigenvalues and the eigenfunctions of the Sturm-Liouville problem:

$$\frac{\mathrm{d}}{d\eta}\left(\eta\frac{\mathrm{d}\Phi}{\mathrm{d}\eta}\right) + \lambda^2\eta\left(1-\eta^N\right)\Phi = 0\,, \tag{16}$$

$$\eta = 0\,,\; \frac{\mathrm{d}\Phi}{\mathrm{d}\eta} = 0\,;\; \eta = 1\,,\; \frac{d\Phi}{d\eta} = 0\,. \tag{17}$$

The Application of Galerkin's Method

For the determination of the eigenfunctions and eigenvalues of the Sturm-Liouville problem (16), (17) we will apply the Galerkin's method. For this, we consider the operators :

$$\begin{gathered} U : D(U) \subset L_2[0,1] \to L_2[0,1], \\ D(U) = \left\{\Phi \in C^2[0,1],\, \frac{\mathrm{d}\Phi}{\mathrm{d}\eta}(0) = 0\,,\, \frac{d\Phi}{d\eta}(1) = 0\right\}, \\ U(\Phi) = \frac{d}{d\eta}\left(\eta\frac{\mathrm{d}\Phi}{\mathrm{d}\eta}\right) + \lambda^2\left(1-\eta^N\right)\Phi\,. \end{gathered} \tag{18}$$

We look at the solution of Sturm-Liouville problem (16), (17) under the approximate form

$$\Phi(\eta) = \sum_{k=1}^{n} a_k \varphi_k(\eta)\,, \tag{19}$$

where $n \in \mathbf{N}^*$ is the approach level of function Φ and $(\varphi_k)_{k\in\mathbf{N}^*}$ is a complete system of functions in $L_2[0,1]$, functions which verifies the conditions [9]

$$\frac{\mathrm{d}\varphi_k}{\mathrm{d}\eta}(0)=0\,,\ \frac{d\varphi_k}{d\eta}(1)=0\,,\ k\in\mathbf{N}^*\,. \tag{20}$$

The unknown coefficients a_k, $k=\overline{1,n}$ are determined if giving the conditions

$$< U(\Phi),\varphi_j >=0\,,\ j=\overline{1,n}\,, \tag{21}$$

the scalar product being considered in the space of square integrable function $L_2[0,1]$.
By applying these conditions, we obtain the linear algebraic system in unknown a_k, $k=\overline{1,n}$:

$$\sum_{k=1}^{n}\left(\alpha_{kj}+\lambda^2\beta_{kj}\right)a_k=0\,,\ j=\overline{1,n}\,, \tag{22}$$

where

$$\alpha_{kj}=\int_0^1\eta\frac{\mathrm{d}\varphi_k}{\mathrm{d}\eta}\frac{d\varphi_j}{\mathrm{d}\eta}\mathrm{d}\eta\,,\ j,k=\overline{1,n}\,, \tag{23}$$

$$\beta_{kj}=\int_0^1\eta\left(1-\eta^N\right)\varphi_k\varphi_j\mathrm{d}\eta\,,\ j,k=\overline{1,n}\,. \tag{24}$$

Because the system (22) must have nontrivial solutions, we obtain the equation

$$\Delta_n\equiv\left|\,A+\lambda^2B\,\right|=0\,, \tag{25}$$

where A and B are the matrix $A=\left(\alpha_{kj}\right)_{k,j=\overline{1,n}}$, $B=\left(\beta_{kj}\right)_{k,j=\overline{1,n}}$.

The solutions of equations (25) represent the approximate values, for the n approach level, for the eigenvalues $\lambda_1^2,\ \lambda_2^2,\Lambda\ ,\lambda_n^2$.

The solution of equation (1) is difficult to be obtained under this form. Consequently, through elementary transformations of determinant Δ_n, this equation takes the form [10]:

$$\left|\,C-\lambda^2I_n\,\right|=0\,, \tag{26}$$

where I_n is the identity matrix of n order.

Unlike matrix A and B which are symmetric, matrix C does not have this property anymore. Therefore we must adopt an adequate method for the determination of its eigenvalues [11].

In the followings, we will use the complete system of functions $(\varphi_k)_{k\in\mathbf{N}^*}$ in $L_2[0,1]$:

$$\varphi_k(\eta)=J_0(\mu_k\eta)\,, \tag{27}$$

where J_0 is the Bessel function of the first kind and zero order and $\mu_k, k\in N^*$ are the roots of the equation:

$$J_1(\mu)=0\,. \tag{28}$$

The integrals which appear in the formulas (23), (24) are calculated with a quadrature formula that must be compatible with Galerkin's method [12]. The eigenvalues of the Sturm-Liouville problem obtained by this method are presented in the next section.

The eigenfunctions of the problem (18), (19) have the analytical form

$$\Phi_i(\eta) = \sum_{j=1}^{n} c_{ij} J_0(\mu_j \eta), \; i = \overline{1,n} \tag{29}$$

where $(c_{i1}, c_{i2}, \Lambda, c_{in}), i = \overline{1,n}$ are the eigenvectors of the matrix $A + \lambda^2 B$.

The Approximate Solution of the Problem

The unknown function u, for the n level of approximation of Galerkin's method, is obtained from (15) and (27):

$$u(\psi, \eta) = \sum_{k=1}^{n} \left(\sum_{i=1}^{n} c_i c_{ik} \mathrm{e}^{-\lambda_i^2 \psi} \right) J_0(\mu_k \eta), \tag{30}$$

The coefficients c_i, $i = \overline{1,n}$ from (30) are determined by the use of the condition (14) and by considering that the solutions $\Phi_i, i = \overline{1,n}$ of the problem (16), (17) are orthogonal with weight $\eta(l - \eta^N)$ on $[0,1]$ [9]. Because the functions $\Phi_i, i = \overline{1,n}$ are not obtained exactly, we prefer to use the orthogonality with weight η of Bessel functions on $[0,1]$.

Thus, for the n level of approximation, the constants c_i, $i = \overline{1,n}$ are determined by the resolution of the linear algebraic system:

$$\sum_{i=1}^{n} c_{ik} c_i = -\frac{N_{Br}}{2N} \frac{\int_0^1 \eta \left(\eta^2 - \frac{1}{N+2} \eta^{N+2} \right) J_0(\mu_k \eta) \mathrm{d}\eta}{\int_0^1 \eta J_0^2(\mu_k \eta) \mathrm{d}\eta}, \quad k = \overline{1,n} \tag{31}$$

The final solution of the problem is obtained now by using the relations (12), (15) and (30):

$$\theta(\psi, \eta) = \frac{N_{Br}}{2N} \left(4\psi + \eta^2 - \frac{2}{N+2} \eta^{N+2} \right) + \sum_{k=1}^{n} \left(\sum_{i=1}^{n} c_i c_{ik} \mathrm{e}^{-\lambda_i^2 \psi} \right) J_0(\mu_k \eta) \quad , \tag{32}$$

As an example, we will consider a fluid with unit Brinkman number. The eigenvalues of the Sturm-Liouville problem (16), (17) are presented in table 1. The coefficients given by (23) and (24) are obtained by a numerical quadrature procedure [11]. The eigenvalues have been obtained by using the procedures BALANC, ELMHES, HQR [11]. The system (31) has been solved using a procedure based on Gauss method [11].

The variation of dimensionless temperature θ given by (32) is presented in figures 1-4. In abscissa axis there is the reduced radial distance η and in ordinates axis there is the dimensionless temperature θ. The variation of dimensionless temperature θ is presented for some values of dimensionless variable ψ.

Given the results obtained, we can deduce that for a certain value of the rheological coefficient *n,* the temperature of the fluid is increased along the tube. For a given value of the dimensionless variable ψ, the temperature of the fluid is increased together with *n*.

The calculations have been realised for the approximation level $n = 10$ and the algorithm presents considerable stability.

Table 1. Eigenvalues of Sturm-Liouville problem

n									
0,35	0,5	0,6	0,7	0,75	0,8	0,9	1,0	1,1	1,2
λ_n^2									
0	0	0	0	0	0	0	0	0	0
20.942	22.427	23.247	23.966	24.293	24.602	25.169	25.679	26.141	26.560
68.283	73.102	75.800	78.175	79.260	80.283	82.168	83.863	85.395	86.788
141.810	151.783	157.384	162.323	164.581	166.712	170.637	174.170	177.366	180.271
241.465	258.414	267.946	276.356	280.201	283.832	290.522	296.544	301.993	306.947
367.227	392.974	407.462	420.251	426.098	431.622	441.799	450.961	459.253	466.792
519.085	555.452	575.924	593.997	602.262	610.070	624.458	637.411	649.135	659.795
697.032	745.840	773.322	797.588	808.685	819.170	838.490	855.885	871.630	885.948
901.064	964.136	999.656	1031.02	1045.36	1058.91	1083.89	1106.38	1126.73	1145.24
1131.17	1210.33	1254.91	1294.29	1312.29	1329.31	1360.66	1388.89	1414.45	1437.69

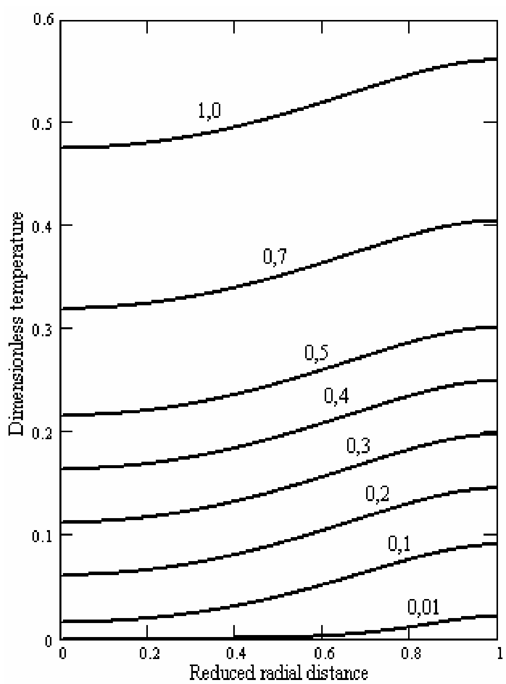

Fig. 1. Dimensionless temperature profiles for adiabatic walls, n=0,35, N_{Br}**=1**

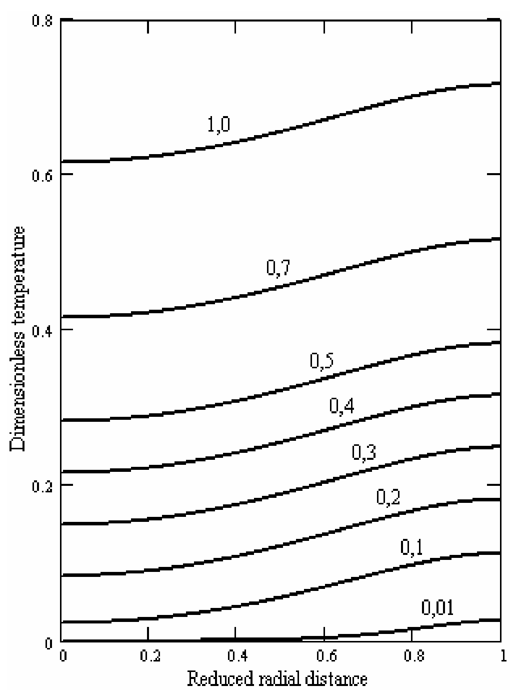

Fig. 2. Dimensionless temperature profiles for adiabatic walls, n=0,5, N_{Br}=1

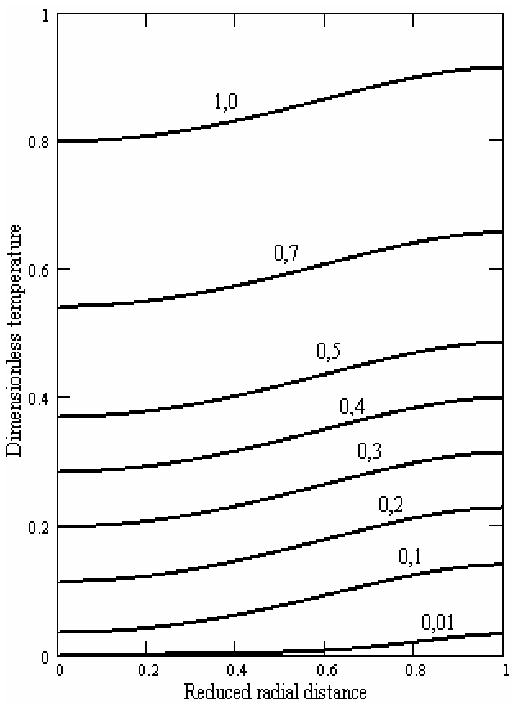

Fig. 3. Dimensionless temperature profiles for adiabatic walls, n=0,75, N_{Br}=1

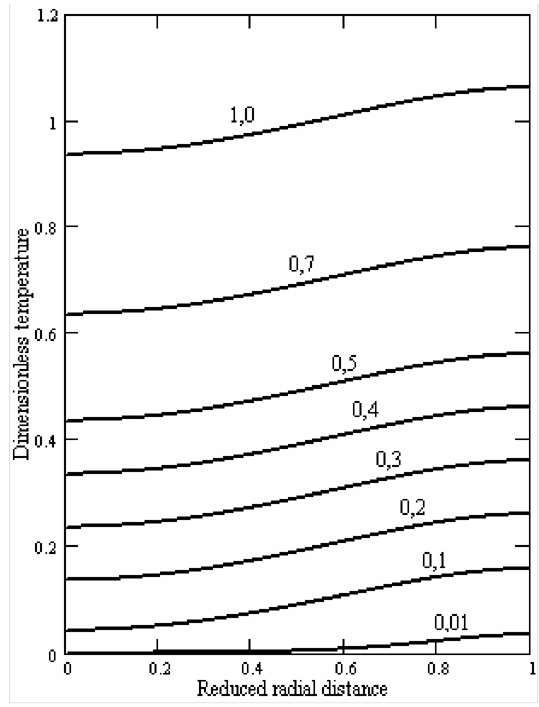

Fig. 4. Dimensionless temperature profiles for adiabatic walls, n=1,0, N_{Br}=1

As compared to Valko [2], the paper presents the advantage of a simpler algorithm which can also be adapted to other boundary conditions (Dirichlet and Robin type conditions) by an appropriate changing of the condition (17) and of the equation (28).

References

1. Y b a r r a, R. M., E c k e r t, R. E., Viscous Heat Generation in Slit Flow, *AiChe Journal*, 26, 5, 1980, pp. 751-762.
2. V a l k ó, P. P., Solution of the Graetz–Brinkman Problem with the Laplace Transform Galerkin Method, *Int. J. Heat Mass Transfer*, 48, 2005**,** pp. 1874-1882.
3. G o t t i f r e d i, J. C., Q u i r o g a, O. D., F l o r e e, A. F., Heat Transfer to Newtonian and non-Newtonian Fluids flowing in a Laminar Regime, *International Journal Heat Mass Transfer*, 26, 1983, pp. 1215-1220.
4. S h i h, Y. P., T s o u, J. D., Extended Leveque Solutions for Heat Transfer to Power-law Fluids in Laminar Flow in a Pipe, *Chem. Eng. J.*, 15, 1978, pp. 55-62.
5. J o h n s t o n, P. R., A Solution Method for the Graetz Problem for non-Newtonian Fluids with Dirichlet and Neumann Boundary Conditions, *Math. Comput. Model*, 19, 1994, pp. 1-19.
6. B o a c ă, T., On Viscous Dissipation in Incompressible Fluid Flow Through a Circular Cross Section Tube, *The Annals of University "Dunărea de Jos" of Galaţi*, Fascicle VIII, Tribology (to appear soon).
7. C o n s t a n t i n e s c u, V. N., *Dinamica fluidelor vâscoase în regim laminar,* Editura Academiei, Bucureşti, 1987.
8. B o a c ă, T., Utilizarea metodei lui Galerkin pentru determinarea valorilor proprii ale problemei Graetz-Nusselt, *St. Cerc. Mec. Apl.*, 53**,** 6, 1994, pp. 549-560.
9. D i n c ă, G., *Metode variaţionale şi aplicaţii,* Editura Tehnică, Bucureşti, 1980.

10. O r o v e a n u, T., S i r o, B., Un algorithme servant à résoudre le problème de Sturm-Liouville par la méthode de Galerkine dans le cas de l'utilisation des systèmes automatiques de calcul, *Rev. Roum. Sci. Techn.-Méc. Appl.*, 30, 2-3, 1985, pp. 161-172.
11. W i l l i a m, H. P., *Numerical Recipes in Pascal,* Cambridge University Press, Cambridge, 1989.
12. Ş c h i o p, Al. I., *Analiza unor metode de* d*iscretizare,* Editura Academiei, Bucureşti, 1978.

Asupra disipației vâscoase în mişcarea fluidelor de tip putere printr-un tub de secțiune circulară

Rezumat

În acest articol este studiată problema disipației vâscoase în mişcarea fluidelor de tip putere printr-un tub de secțiune circulară, cu condiții la limită de tip Neumann. Soluția problemei este obținută sub forma unei serii după sistemul complet de funcții proprii al unei probleme de tip Sturm-Liouville. Valorile proprii şi funcțiile proprii ale acestei probleme Sturm-Liouville sunt obținute cu metoda lui Galerkin.