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The Influence of the Shear Force in the Deflections of Cantilevers with Variable Cross Sectional Area

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Abstract

In the paper is analysed the influence of the shear force in the deflections of cantilevers loaded with an uniform distributed external pressure. The differential equation of the elastic curve of the beam is integrated in two hypothesis: when the influence of the shear force in the deflection is neglected and when is not. In both cases it is obtained the analytical solution of the deflections and slopes. The results obtained are analysed in a calculus example.

Key words: *deflection, slope, elastic curve, shear force*

General Equations

A cantilever with a variable cross sectional area is considered (fig. 1). The beam has a linear depth b_x, the length *l,* the thickness *h* and is externally loaded with an uniform pressure *q*.

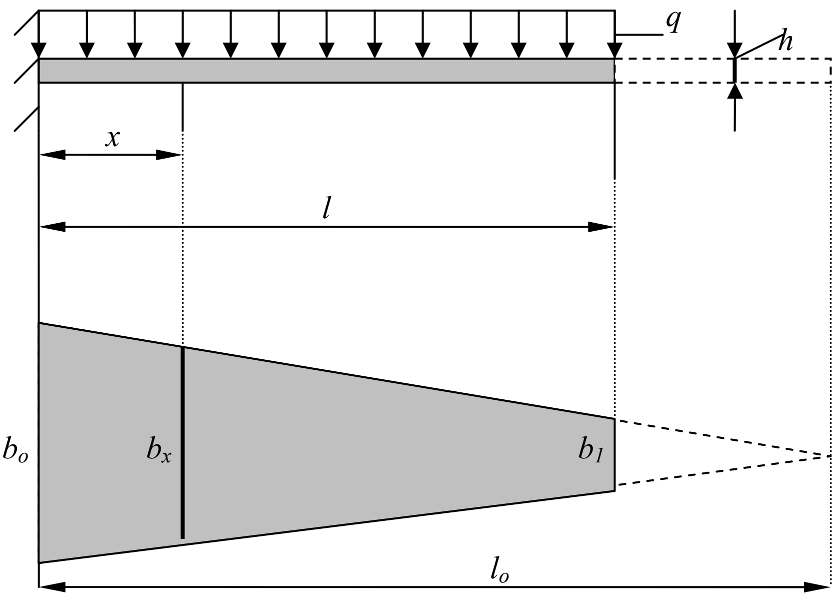

Fig.1. A cantilever loaded with uniform pressure

The beam is embedded at the left end and its depth presents a linear variation along the length. If the depth in the origin is b_o the current depth of the beam can be expressed from the geometry of the beam as $b_x = b_o \frac{l_o - x}{l_o}$, where l_o represents the entire length of the triangle that the beam is extracted from. The main geometrical characteristics of the cross sectional area (the inertia moment and the area) can be calculated with the relations:

$$I_z(x) = b_x \frac{h^3}{12} = b_o \frac{l_o - x}{l_o} \cdot \frac{h^3}{12} \tag{1}$$

$$A(x) = b_x \cdot h = \frac{b_o \cdot h}{l_o}(l_o - x) \tag{2}$$

The area of the cross surface of the beam where the tangential stresses (τ_{xy}) are considered to be uniformly distributed can be calculated with the relation [1]:

$$A'(x) = \frac{5}{6} A(x) = \frac{5}{6} \frac{b_o h}{l_o}(l_o - x) \tag{3}$$

The bending moment in the current section of the beam can be expressed by reducing all the moments from the left side:

$$M_z(x) = -\frac{q \cdot l^2}{2} + q \cdot l \cdot x - q\frac{x^2}{2} \tag{4}$$

The second order differential equation that describes the bending of the elastic curve of the beam when the influence of the shear force is neglected can be written under the form:

$$\frac{d^2 v}{dx^2} = -\frac{M_z(x)}{EI_z(x)} = \frac{\frac{q}{2}(l - x)^2}{E\frac{b_o h^3}{12}\frac{l_o - x}{l_o}} = \frac{6 \cdot q \cdot l_o}{E \cdot b_o h^3}\frac{(l - x)^2}{l_o - x} = A_o \frac{(l - x)^2}{l_o - x} \;, \tag{5}$$

where A_o is a constant that can be identified from (5) and has the following form:

$$A_o = \frac{6 \cdot q \cdot l_o}{E \cdot b_o \cdot h^3} \tag{6}$$

In order to obtain the expressions of the slope(φ) and deflection(v) the above differential equation has to be integrated step by step in respect with the current variable x, taking into consideration the limit conditions ($x = 0 \Rightarrow \varphi = 0$ and $v = 0$):

$$\varphi(x) = A_o \cdot x\left(2l - l_o - \frac{x}{2}\right) + A_o \cdot (l_o - l)^2 \cdot \ln\frac{l_o}{l_o - x} \tag{7}$$

$$v(x) = -A_o \cdot \frac{x^3}{6} + A_o \cdot \frac{x^2}{2} \cdot (2l - l_o) + A_o (l_o - l)^2 x + A_o \cdot (l_o - l)^2 \cdot (l_o - x) \cdot \ln \frac{l_o - x}{l_o} \quad (8)$$

The maximum deflection of the beam is reached in the free end and can be obtained from (8) when $x = l$:

$$v_{\max} = A_o \cdot l \cdot \left(l_o^2 - \frac{5}{2} l_o l + \frac{11}{6} l^2 \right) + A_o \cdot (l_o - l)^3 \ln \frac{l_o - l}{l_o} \quad (9)$$

The second order differential equation of the elastic curve of the beam when the influence of the shear force is considered has the form [1]:

$$\frac{d^2 v}{dx^2} = -\frac{M_z(x)}{E \cdot I_z(x)} - \frac{q}{G \cdot A'(x)} \quad (10)$$

Replacing (4), (1) and (3) in (10) the following differential equation is obtained:

$$\frac{d^2 v}{dx^2} = A_o \frac{(l - x)^2}{l_o - x} - B_o \frac{1}{l_o - x} \quad , \quad (11)$$

where B is a constant that, for a rectangle surface (in respect with 3) has the form:

$$B_o = \frac{6 \cdot q \cdot l_o}{5 \cdot b_o \cdot h \cdot G} \quad (12)$$

Integrating twice the above differential equation (in respect with x) and taking into consideration that at the embedded end the slope and deflection have to be null ($x = 0 \Rightarrow \varphi_1 = 0; v_1 = 0$), the following expressions for displacements are obtained:

$$\varphi_1(x) = A_o \cdot x \left(2l - l_o - \frac{x}{2} \right) + A_o \cdot (l_o - l)^2 \cdot \ln \frac{l_o}{l_o - x} + B_o \cdot \ln \frac{l_o - x}{l_o} \quad (13)$$

$$v_1(x) = -A_o \cdot \frac{x^3}{6} + A_o \cdot \frac{x^2}{2} \cdot (2l - l_o) + A_o (l_o - l)^2 x + A_o \cdot (l_o - l)^2 \cdot (l_o - x) \cdot \ln \frac{l_o - x}{l_o} + \\ + B_o (l_o - x) \cdot \ln \frac{l_o}{l_o - x} - B_o x \quad (14)$$

From (14) it can be expressed the maximum deflection of the beam (in the free edge):

$$v_{1,\max} = A_o \cdot l \cdot \left(l_o^2 - \frac{5}{2} l_o l + \frac{11}{6} l^2 \right) + A_o \cdot (l_o - l)^3 \ln \frac{l_o - l}{l_o} + B_o (l_o - l) \ln \frac{l_o}{l_o - l} - B_o l \quad (15)$$

Analysing the above expression it can be noticed that the last two terms from (15) represents the influence of the shear force in the maximum deflection of the beam.

In order to evaluate the relative influence of the shear force in the deflection of the beam it may be calculate the raport :

$$Er = \frac{v_{\max} - v_{1,\max}}{v_{\max}} = \frac{B_o}{A_o} \cdot \frac{l - (l_o - l) \cdot \ln \dfrac{l_o}{l_o - l}}{l\left(l_o^2 - \dfrac{5}{2} l_o l + \dfrac{11}{6} l^2\right) + (l_o - l)^3 \cdot \ln \dfrac{l_o - l}{l_o}} \tag{16}$$

Replacing (6) and (12) into (16) the following expression of the relative influence of the shear force is obtained:

$$Er = \frac{h^2}{2} \cdot \frac{l - (l_o - l) \cdot \ln \dfrac{l_o}{l_o - l}}{l\left(l_o^2 - \dfrac{5}{2} l_o l + \dfrac{11}{6} l^2\right) + (l_o - l)^3 \cdot \ln \dfrac{l_o - l}{l_o}} \tag{17}$$

A Numerical Example

In order to evaluate numerically the influence of the shear force in the maximum bending deflection, a beam with the following geometrical characteristics is considered: l_o = 1 m and h = 80 mm. It can be noticed that the coefficient h/l is smaller than 0.1 for lengths higher than 800 mm, that means that in the classical theory of bending of beams with a constant cross sectional area the influence of the shear force can be neglected.

In our case (beam with variable cross sectional area) the expression (17) has been graphically represented for different values of l (fig. 2).

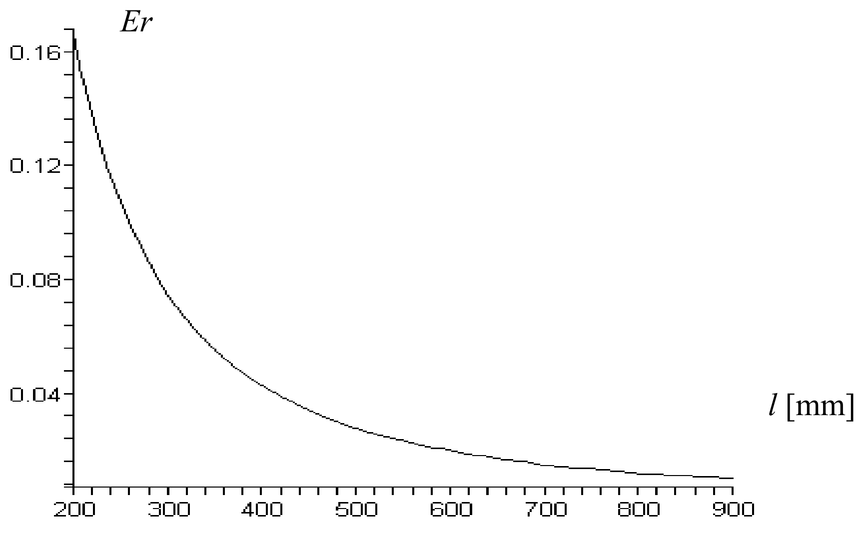

Fig. 2. The variation of the relative influence against l for h = 80 mm

Analysing the above graphical representation it can be noticed that it exist an interval (for $l \leq 450$ mm) where the influence of the shear force is higher than 5% and has to be considered.

Such variations as those presented in figure 2 has been obtained for different thicknesses of the beam and are presented in figures 3 (for h = 100 mm), 4(for h = 120 mm) and 5 (for h = 140 mm). All this diagrams shows that as thickness is getting higher as the area where the influence of the shear force is significant is more extended.

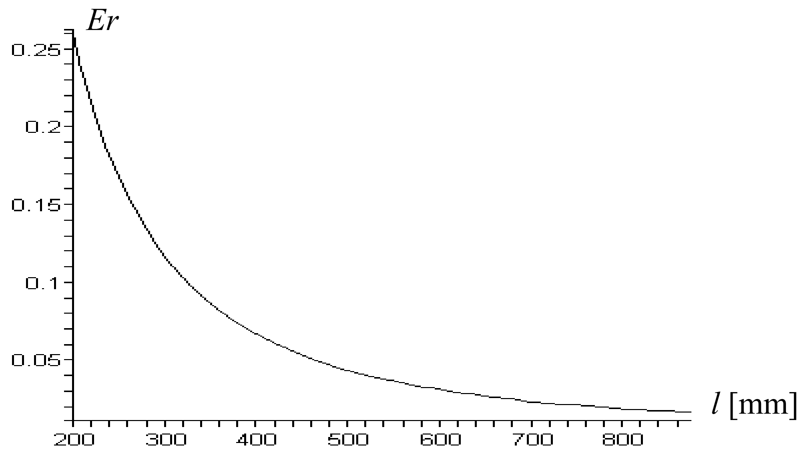

Fig. 3. The variation of the relative influence against l for h = 100 mm

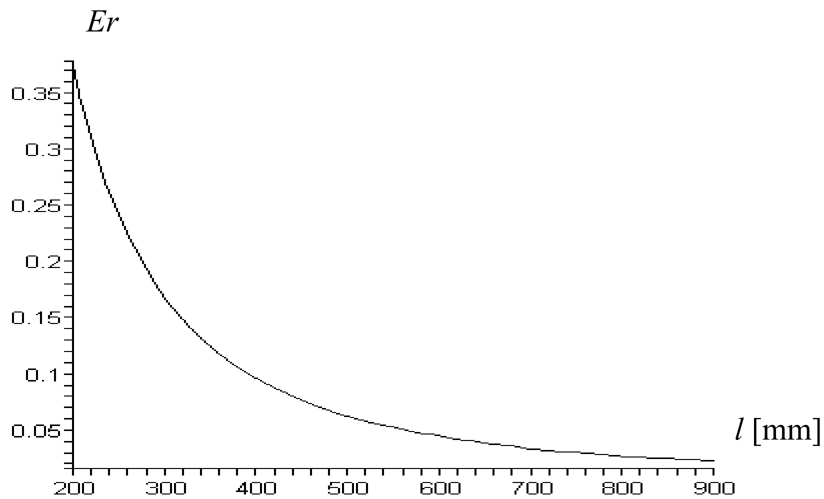

Fig. 4. The variation of the relative influence against l for h = 120 mm

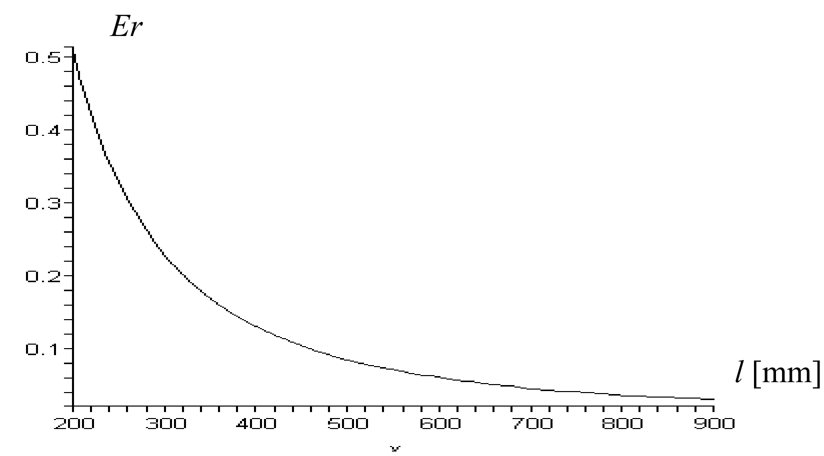

Fig. 5. The variation of the relative influence against l for h = 140 mm

From the figures 3, 4 and 5 it can be noticed that the influence of the shear force has to be considered for a thickness of the beam higher tham 120 mm. For h = 120 mm and the length of the beam l < 600 mm the influence of the shear force is higher than 5% (fig. 4). For a higher thickness (h =140 mm) the shear effect is overpassing 15% for a length of a beam l < 400 mm..

Conclusions

In the paper is presented a secondarry effect (that can be ussually neglected) for a cantilever beam with a variable cross sectional area. From the results obtained it can be noticed that it exists a range of dimensions (length and thickness of the beam) where the influence of the shear force in the deflection of the beam becomes significant and is recommended to be considered or analysed.

References

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Influența forței tăietoare în deformarea consolelor cu secțiune variabilă

Rezumat

În lucrare se analizează influența forței tăietoare asupra consolelor cu secțiunea variabilă. Ecuația diferențială a fibrei medii deformate este integrată in două ipoteze: când influența forței tăietoare este neglijată şi când este luată în considerare. În ambele cazuri sunt obținute expresiile rotirilor şi săgeților. Rezultatele obținute sunt analizate pe un exemplu de calcul.