| BULETINUL Universităţii Petrol – Gaze din Ploieşti | Vol. LXIII No. 3/2011 | 35 - 38 | Seria Tehnică |
|---|---|---|---|

# Energetical Aspects Regarding the Extra-Supported Structures

Vasilii Graur, Şerban Vasilescu, Dorin Bădoiu

Universitatea Petrol-Gaze din Ploieşti, Bd. Bucureşti, 39, Ploieşti
e-mail: vserban@mail.upg-ploiesti.ro

## Abstract

*In the paper it is presented a way of solving the extra-supported structures using the theorem of minimum of the potential energy. In this respect the potential energy is expressed function of unknown reactions (forces and moments) and the minimum conditions are used. From this condition results the forces and moments from the supported or embedded points. This way of calculation is exemplified in some classical examples.*

**Key words:** *potential energy, reactions, deflection*.

## General Aspects

For a plane structure that is subjected to bending the potential energy can be expressed with the classical relation [1] :

$$U = \int \frac{M^2(x)}{2 \cdot EI} dx \quad , \tag{1}$$

where $M(x)$ represents the bending moment, $EI$ is the bending rigidity of the beam and the integral covers the all lengths of the beams that the structures is made from.

In order to solve an extra-supported beam it is necessary to express the potential energy from (1) in function of the unknown reactions (forces and moments) and to use the minimum conditions that require that the first derivative in respect with the unknowns to be zero. It is also important to make sure that the solutions of the above conditions reach the minimum of the potential energy. This condition is accomplished when the second derivative in respect with the unknown is positive or a similar condition presented in [2] is also positive.

In the case that the potential energy contains only a variable $R$ the minimum conditions are:

$$\frac{dU}{dR} = 0 \quad \text{a)} \quad \text{and} \quad \frac{d^2U}{dR^2} > 0 \quad \text{b)} \tag{2}$$

In the case that the potential energy is a function of two variables ($R_1$ and $R_2$), the minimum condition are [2]:

$$\frac{\partial U}{\partial R_1}=0 \quad a) \qquad \frac{\partial U}{\partial R_2}=0 \quad b) \qquad \frac{\partial^2 U}{\partial R_1^2}\cdot\frac{\partial^2 U}{\partial R_2^2}-\left(\frac{\partial^2 U}{\partial R_1 \partial R_2}\right)^2>0 \quad c) \tag{3}$$

## First Example

It is considered a beam symmetrically supported in three points and loaded with two concentrated forces (fig.1).

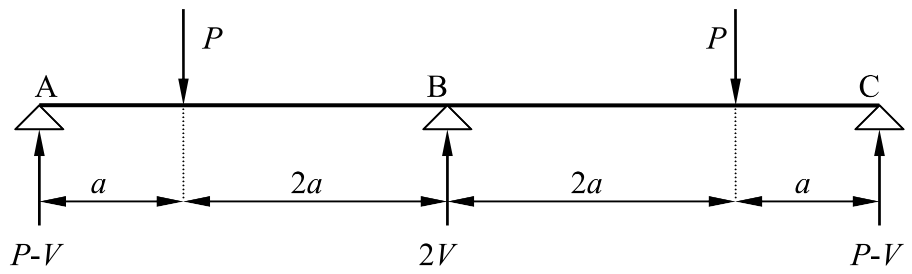


**Fig.1.** A beam supported in three points

The beam is also symmetrically loaded and in order to express the potential energy is necessary to find the function of the bending moment from the current section. For the first segment A-B the function of the bending moment can be written under the form:

$$M(x)=\begin{cases}(P-V)x, x\in[0,a) \\ P\cdot a - V\cdot x, x\in[a,3a)\end{cases} \tag{4}$$

Using the (1) relation the potential energy for the whole beam can be expressed taking into consideration the symmetry of the beam:

$$U(V)=2\left[\int_0^a \frac{(P-V)^2 x^2}{2EI}dx+\int_a^{3a}\frac{(P\cdot a - V\cdot x)^2}{2EI}dx\right]= \\ =\frac{(P-V)^2 a^3}{3EI}+\frac{2P^2a^3}{EI}-\frac{8PVa^3}{EI}+\frac{26\cdot V^2 a^3}{3EI} \tag{5}$$

In order to find the unknown reaction $V$ is necessary to use the minimum condition (2):

$$\frac{dU}{dV}=-\frac{2(P-V)a^3}{3EI}-\frac{8Pa^3}{EI}+\frac{52\cdot V\cdot a^3}{3EI}=0 \Rightarrow V=\frac{13}{27}P \tag{6}$$

Once the reaction $V$ is calculated (from energetical considerations) it can be also calculated the end reactions:

$$V_A=P-V=P-\frac{13}{27}P=\frac{14}{27}P \tag{7}$$

It can be noticed that the energetical considerations allow the calculation of the reactions from the supported points easier that the classical way of solving the extra-supported beams. In order to verify the values of the reactions it can be find the values of deflections in some points where these values are already known.

It can also be noticed that the second derivative of the potential energy is positive ($d^2U/dV^2 = 18a^3/EI > 0$).

For example it can be calculated the deflection in point C (that has to be zero). In this respect the relation of deflection in the origin parameters formulation can be used. From the condition that the deflection in B is zero can be expressed the slope:

$$v_B = \varphi_o \cdot 3a - \frac{\frac{14}{27}P}{6EI}(3a)^3 + \frac{P}{6EI}(2a)^3 = 0 \Rightarrow \varphi_o = \frac{Pa^2}{3EI}$$

With the initial slope it can be calculated the deflection in any point. For example the deflection in C is:

$$v_C = \frac{Pa^2}{3EI}(6a) - \frac{\frac{14}{27}P}{6EI}(6a)^3 + \frac{P}{6EI}(5a)^3 - \frac{\frac{26}{27}P}{6EI}(3a)^3 + \frac{P}{6EI}a^3 = 0$$

It can be noticed that the deflection in C has an expected value that means that the reactions of the extra-supported beam are correct.

## Another Example

A beam embedded at the ends, simply supported in the middle and loaded externally with two symmetric concentrated forces is considered (fig. 2).

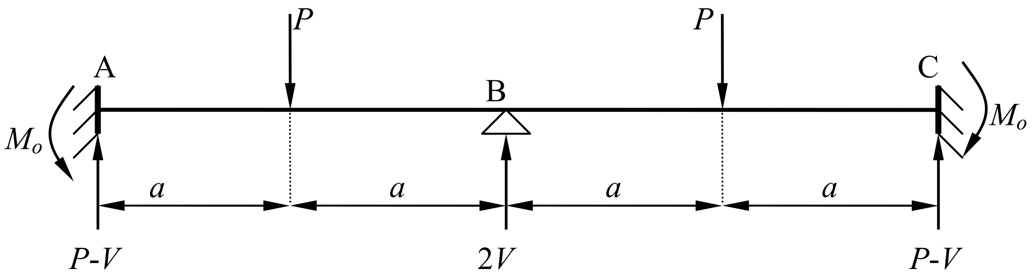


**Fig.2** A beam embedded at the ends and simply supported in the middle

In order to calculate the unknown reactions ($V$ and $M_o$) it is necessary to follow the same steps like in the above example. The expression of the bending moment for the A-B segment has the form:

$$M(x) = \begin{cases} -M_o + (P - V)x, x \in [0, a) \\ -M_o - V \cdot x + P \cdot a, x \in [a, 2a) \end{cases} \quad (8)$$

The potential energy for the whole beam can be expressed considering the relation (2) under the form:

$$U(V,M_o)=2\left[\int_0^a \frac{((P-V)x-M_o)^2}{2EI}dx+\int_a^{2a}\frac{(Pa-M_o-Vx)^2}{2EI}dx\right]=$$
$$=\frac{(P-V)^2}{3EI}a^3-\frac{M_o(P-V)}{EI}a^2+\frac{M_o^2}{EI}a+\frac{(Pa-M_o)^2}{EI}a-\frac{3(Pa-M_o)\cdot V}{EI}a^2+\frac{7\cdot V^2}{3EI}a^3 \tag{9}$$

In order to find the values for the unknowns ($V$ and $M_o$) the minimum conditions (3) had to be used:

$$\frac{\partial U}{\partial V}=\frac{-2(P-V)}{3EI}a^3+\frac{M_o a^2}{EI}-\frac{3(Pa-M_o)}{EI}a^2+\frac{14\cdot V}{3EI}a^3=0 \qquad \text{a)}$$
$$\frac{\partial U}{\partial M_o}=-\frac{P-V}{EI}a^2+\frac{2M_o}{EI}a-\frac{2(Pa-M_o)}{EI}a+\frac{V}{EI}3a^2=0 \qquad \text{b)} \tag{10}$$

Solving the above equations system it results: $M_o=\frac{Pa}{4}$, $V=\frac{P}{2}$. Verifying the minimum conditions it results:

$$\frac{\partial^2 U}{\partial V^2}\cdot\frac{\partial^2 U}{\partial^2 M_o}-\left(\frac{\partial^2 U}{\partial V\partial M_o}\right)^2=\frac{16a^3}{3EI}\cdot\frac{4a}{EI}-\left(\frac{4a^2}{EI}\right)^2=\frac{16a^4}{3(EI)^2}>0 \quad , \tag{11}$$

that proves the minimum requirements for the potential energy.

## Conclusions

In the paper is presented a way of solving the extra-supported structures using some energetical consideration. The results obtained are exemplified in some classical models.

## References

1. P o s e a N . – *Rezistenţa materialelor*, Editura Didactică şi Pedagogică, Bucureşti, 1979;
2. F o r r a y M . J . - *Calculul variaţional in stiinta si tehnică,* Editura Tehnică, Bucuresti, 1975.

# Aspecte energetice privind sistemele „supra-legate”

## Rezumat

*În lucrare se prezintã o metodã de rezolvare a sistemelor static nedeterminate utilizând considerente energetice. Astfel, impunând condiţia de minim a energiei potenţiale se determină reacţiunile necunoscute. Aceste rezultate sunt exemplificate pe câteva modele clasice.*