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# On the Establishing and Solving of the Movement Equation in the Case of the Plane Mechanisms

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## Abstract

*In the paper a method for establishing and solving the movement equation in the case of the plane mechanisms is presented. The method is based on the expressing of the dynamic equilibrium in instantaneous powers of all the forces and moments that work on the component links of the mechanism. The variation of the angular speed of the driving crank is determined by numerically integration of the movement equation using the finite differences method. Finally, some simulation results in the case of a quadrilateral mechanism are shown.*

**Key words:** *plane mechanism, dynamic equilibrium, movement equation*

## Introduction

The setting of the dynamic response of the mechanisms for different functioning regimes and loading states represents one of the most important tasks in the phase of their design. In the case of the plane mechanisms having the degree of mobility equal to one, when the leader element is a crank, the setting of the dynamic response involves the establishing and the solving of the dynamic movement equations of these mechanisms [1]. By solving the movement equations the variation of the angular speed of the driving cranks can be determined. It is known that if in the regime phase the angular speed of the driving crank has large variations around its nominal value then appear significant additional variable loads of the links and of the joints of the mechanism [2, 3].

In this paper a method for establishing and solving the movement equation in the case of the plane mechanisms with one degree of mobility, when the leader element is a crank, is presented. The method is based on the expressing of the dynamic equilibrium in instantaneous powers of all the forces and moments that work on the component links of the mechanism. The variation of the angular speed of the driving crank is determined by numerically integration of the movement equation using the finite differences method.

## Theoretical Considerations and Verification Results

In Figure 1 the cinematic scheme of a plane mechanism with one degree of mobility, when the leader element is a crank, is presented. $\varphi_1$ is the driving crank angle; $\omega_1$ and $\varepsilon_1$ are the angular speed and the angular acceleration, respectively, of the driving crank; $C_j, j=\overline{1,n}$ are the mass centers of the component links; $M_m$ is the motor moment; $\overline{F}_j$ and $\overline{M}_j, j=\overline{1,n}$, are the resultant forces and the resultant moments, respectively, corresponding to the external forces and moments (weight forces, forces and moments of technological resistance) that work on the component elements $j, j=\overline{1,n}$; $\overline{F}_{ij}=-m_j\cdot\overline{a}_{C_j}$, and $\overline{M}_{ij}=-I_{C_j}\cdot\overline{\varepsilon}_j$, $j=\overline{1,n}$, are the resultant inertial forces and moments corresponding to the component elements $j, j=\overline{1,n}$, where: $m_j$ is the mass of the *j* element; $I_{C_j}$ is the mass moment of inertia corresponding to the *j* element; $\overline{a}_{C_j}$ is the acceleration of the mass center of the *j* element; $\overline{\varepsilon}_j$ is the angular acceleration of the *j* element.

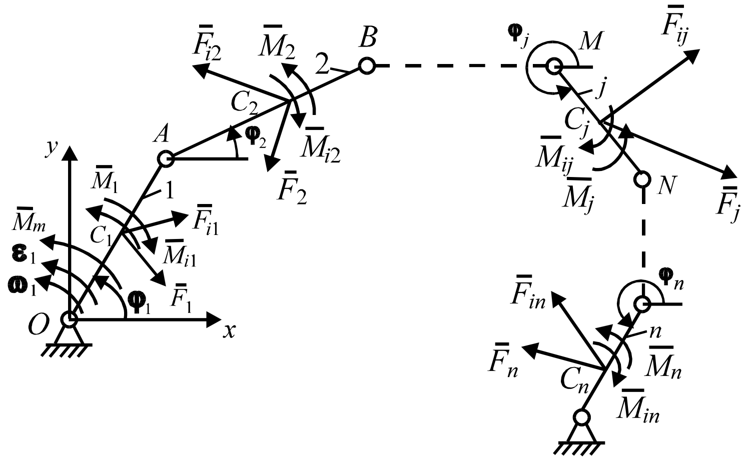


**Fig. 1.** Plane mechanism with one degree of mobility

The dynamic equilibrium in instantaneous powers of all the forces and moments that work on the component links of the mechanism can be expressed with the following relation:

$$\overline{M}_m\cdot\overline{\omega}_1+\sum_{j=1}^{n}(\overline{F}_j\cdot\overline{v}_{C_j}+\overline{M}_j\cdot\overline{\omega}_j)+\sum_{j=1}^{n}(\overline{F}_{ij}\cdot\overline{v}_{C_j}+\overline{M}_{ij}\cdot\overline{\omega}_j)=0 \quad (1)$$

The components of the speeds $\overline{v}_{C_j}, j=\overline{1,n}$, on *x* and *y* axes and the angular speeds $\omega_j, j=\overline{2,n}$ can be calculated by deriving with respect to time the variation functions corresponding to the coordinates of the mass centers $x_{C_j}, y_{C_j}, j=\overline{1,n}$, and to the angles $\varphi_j, j=\overline{2,n}$ (fig. 1), respectively, with the following relations [4,5]:

$$\begin{cases}(v_{C_j})_x=\dot{x}_{C_j}=\dfrac{\mathrm{d}x_{C_j}}{\mathrm{d}\varphi_1}\cdot\dfrac{\mathrm{d}\varphi_1}{\mathrm{d}t}=\omega_1\cdot\dfrac{\mathrm{d}x_{C_j}}{\mathrm{d}\varphi_1}\\[2ex](v_{C_j})_y=\dot{y}_{C_j}=\dfrac{\mathrm{d}y_{C_j}}{\mathrm{d}\varphi_1}\cdot\dfrac{\mathrm{d}\varphi_1}{\mathrm{d}t}=\omega_1\cdot\dfrac{\mathrm{d}y_{C_j}}{\mathrm{d}\varphi_1}\end{cases} \quad (2)$$

$$\omega_j=\dot{\varphi}_j=\frac{\mathrm{d}\varphi_j}{\mathrm{d}\varphi_1}\cdot\frac{\mathrm{d}\varphi_1}{\mathrm{d}t}=\omega_1\cdot\frac{\mathrm{d}\varphi_j}{\mathrm{d}\varphi_1} \quad (3)$$

The accelerations $\overline{a}_{C_j}, j=\overline{1,n}$, of the mass centers $C_j, j=\overline{1,n}$, and the angular accelerations

$\varepsilon_j, j=\overline{2,n}$, of the component elements $j, j=\overline{2,n}$, that appear in the expressions of the resultant inertial forces $\overline{F}_{ij}$ and of the inertial moments $\overline{M}_{ij}$, respectively, can be calculated with the following relations, obtained by deriving with respect to time the variation functions in relations (2) and (3):

$$\begin{cases} (a_{C_j})_x = \ddot{x}_{C_j} = \varepsilon_1 \cdot \dfrac{\mathrm{d}x_{C_j}}{\mathrm{d}\varphi_1} + \omega_1^2 \cdot \dfrac{\mathrm{d}^2 x_{C_j}}{\mathrm{d}\varphi_1^2} \\ (a_{C_j})_y = \ddot{y}_{C_j} = \varepsilon_1 \cdot \dfrac{\mathrm{d}y_{C_j}}{\mathrm{d}\varphi_1} + \omega_1^2 \cdot \dfrac{\mathrm{d}^2 y_{C_j}}{\mathrm{d}\varphi_1^2} \end{cases} \tag{4}$$

$$\varepsilon_j = \ddot{\varphi}_j = \varepsilon_1 \cdot \frac{\mathrm{d}\varphi_j}{\mathrm{d}\varphi_1} + \omega_1^2 \cdot \frac{\mathrm{d}^2\varphi_j}{\mathrm{d}\varphi_1^2} \tag{5}$$

By considering that the motor moment $M_m$ varies depending on $\omega_1$ according to the following relation:

$$M_m = A - B \cdot \omega_1 \tag{6}$$

where: $A$ and $B$ are two constants whose values depend on the type of the motor and the transmission used and by taking into account the relations (2), (3), (4) and (5), the relation (1) becomes:

$$\begin{aligned} A - B\cdot\omega_1 + \sum_{j=1}^{n}\left(F_{jx}\cdot\frac{\mathrm{d}x_{C_j}}{\mathrm{d}\varphi_1} + F_{jy}\cdot\frac{\mathrm{d}y_{C_j}}{\mathrm{d}\varphi_1} + M_j\cdot\frac{\mathrm{d}\varphi_j}{\mathrm{d}\varphi_1}\right) + \\ + \sum_{j=1}^{n}\left(-m_j\cdot\ddot{x}_{C_j}\cdot\frac{\mathrm{d}x_{C_j}}{\mathrm{d}\varphi_1} - m_j\cdot\ddot{y}_{C_j}\cdot\frac{\mathrm{d}y_{C_j}}{\mathrm{d}\varphi_1} - I_{C_j}\cdot\ddot{\varphi}_j\cdot\frac{\mathrm{d}\varphi_j}{\mathrm{d}\varphi_1}\right) = 0 \end{aligned} \tag{7}$$

where: $F_{jx}$ and $F_{jy}$ are the components of the force $\overline{F}_j$ on $x$ and $y$ axes, respectively.

The movement equation (7) has been numerically integrated using the finite differences method [1,5,6]. In this case the infinitesimal differences $\mathrm{d}\varphi_1$ and $\mathrm{d}\omega_1$ have been replaced by the following finite differences:

$$\begin{cases} \Delta\varphi_1 = \varphi_{1,i+1} - \varphi_{1,i} \\ \Delta\omega_1 = \omega_{1,i+1} - \omega_{1,i} \end{cases} \tag{8}$$

where: $\varphi_{1,i}$ and $\varphi_{1,i+1}$ are the values of the angle $\varphi_1$ for two successive positions of the driving crank and $\omega_{1,i}$ and $\omega_{1,i+1}$ represent the values of the angular speed $\omega_1$ corresponding to the angles $\varphi_{1,i}$ and $\varphi_{1,i+1}$.

Considering that:

$$\varepsilon_1 = \frac{\mathrm{d}\omega_1}{\mathrm{d}t} = \frac{\mathrm{d}\omega_1}{\mathrm{d}\varphi_1}\cdot\frac{\mathrm{d}\varphi_1}{\mathrm{d}t} = \frac{\mathrm{d}\omega_1}{\mathrm{d}\varphi_1}\cdot\omega_1 \tag{9}$$

then the value $\varepsilon_{1,i}$ of the angular acceleration $\varepsilon_1$ corresponding to the angle $\varphi_{1,i}$ can be calculated with the following relation:

$$\varepsilon_{1,i} = \frac{\omega_{1,i+1} - \omega_{1,i}}{\Delta\varphi_1} \cdot \omega_{1,i} \tag{10}$$

By introducing the following notations for the values of the derivatives of the functions $x_{C_j}(\varphi_1), y_{C_j}(\varphi_1)$ and $\varphi_j(\varphi_1)$ corresponding to the angle $\varphi_{1,i}$:

$$\begin{cases} \left(\dfrac{\mathrm{d}x_{C_j}}{\mathrm{d}\varphi_1}\right)_{\varphi_1=\varphi_{1,i}} = t_{ij}; & \left(\dfrac{\mathrm{d}^2 x_{C_j}}{\mathrm{d}\varphi_1^2}\right)_{\varphi_1=\varphi_{1,i}} = tt_{ij} \\ \left(\dfrac{\mathrm{d}y_{C_j}}{\mathrm{d}\varphi_1}\right)_{\varphi_1=\varphi_{1,i}} = p_{ij}; & \left(\dfrac{\mathrm{d}^2 y_{C_j}}{\mathrm{d}\varphi_1^2}\right)_{\varphi_1=\varphi_{1,i}} = pp_{ij} \\ \left(\dfrac{\mathrm{d}\varphi_j}{\mathrm{d}\varphi_1}\right)_{\varphi_1=\varphi_{1,i}} = q_{ij}; & \left(\dfrac{\mathrm{d}^2 \varphi_j}{\mathrm{d}\varphi_1^2}\right)_{\varphi_1=\varphi_{1,i}} = qq_{ij} \end{cases} \tag{11}$$

then the values of the derivatives $\ddot{x}_{C_j}$, $\ddot{y}_{C_j}$ and $\ddot{\varphi}_j$ (in relations (4) and (5)) corresponding to the angle $\varphi_{1,i}$ can be calculated with the following relations:

$$\begin{cases} (\ddot{x}_{C_j})_{\varphi_1=\varphi_{1,i}} = \dfrac{\omega_{1,i+1} - \omega_{1,i}}{\Delta\varphi_1} \cdot \omega_{1,i} \cdot t_{ij} + \omega_{1,i}^2 \cdot tt_{ij} \\ (\ddot{y}_{C_j})_{\varphi_1=\varphi_{1,i}} = \dfrac{\omega_{1,i+1} - \omega_{1,i}}{\Delta\varphi_1} \cdot \omega_{1,i} \cdot p_{ij} + \omega_{1,i}^2 \cdot pp_{ij} \\ (\ddot{\varphi}_{C_j})_{\varphi_1=\varphi_{1,i}} = \dfrac{\omega_{1,i+1} - \omega_{1,i}}{\Delta\varphi_1} \cdot \omega_{1,i} \cdot q_{ij} + \omega_{1,i}^2 \cdot qq_{ij} \end{cases} \tag{12}$$

By taking into account the relations (10), (11) and (12) in the evaluation corresponding to the angle $\varphi_{1,i}$ of the terms that appear in the movement equation (7) the following equation is obtained:

$$A - B \cdot \omega_{1,i} + S_{1,i} + \frac{\omega_{1,i+1} - \omega_{1,i}}{\Delta\varphi_1} \cdot \omega_{1,i} \cdot S_{2,i} + \omega_{1,i}^2 \cdot S_{3,i} = 0 \tag{13}$$

where:

$$S_{1,i} = \sum_{j=1}^{n} (F_{jx.i} \cdot t_{ij} + F_{jy.i} \cdot p_{ij} + M_{j.i} \cdot q_{ij}) \tag{14}$$

$$S_{2,i} = \sum_{j=1}^{n} (-m_j \cdot t_{ij}^2 - m_j \cdot p_{ij}^2 - I_{C_j} \cdot q_{ij}^2) \tag{15}$$

$$S_{3,i} = \sum_{j=1}^{n} (-m_j \cdot t_{ij} \cdot tt_{ij} - m_j \cdot p_{ij} \cdot pp_{ij} - I_{C_j} \cdot q_{ij} \cdot qq_{ij}) \tag{16}$$

In relation (14), $F_{jx.i}$, $F_{jy.i}$ and $M_{j.i}$ are the values of the forces $F_{jx}$ and $F_{jy}$ and of the moment $M_j$, respectively, corresponding to the angle $\varphi_{1,i}$.

Starting from the equation (13) the following computing recursive relation is obtained:

$$\omega_{1,i+1} = \omega_{1,i} + \frac{\Delta\varphi_1}{\omega_{1,i} \cdot S_{2,i}} \cdot (-A + B \cdot \omega_{1,i} - S_{1,i} - \omega_{1,i}^2 \cdot S_{3,i}) \qquad (17)$$

The method presented has been applied in the case of a quadrilateral mechanism (fig. 2). To this end a simulation program has been developed using the Maple programming language that has powerful symbolic computation functions [7].

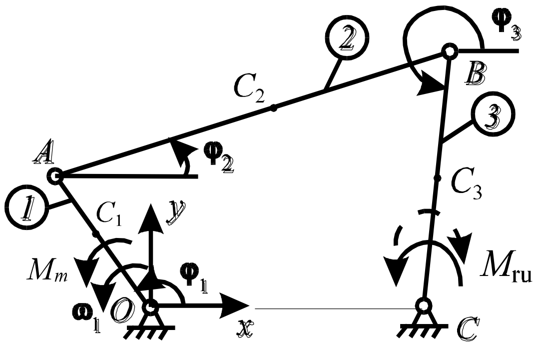


**Fig. 2.** Quadrilateral mechanism

The following elements are considered to be known:

- the dimensions of the component links: *OA*=0.2 m; *AB*=0.5 m; *BC*=0.55 m; *OC*=0.75 m. The mass centers: $C_1, C_2, C_3$ are on the middle of the corresponding links;
- the component links are of bar type, with constant cross section. There are known the values of the linear mass of the links: $q_1 = 5.5\,\text{kg/m}$, $q_2 = 4\,\text{kg/m}$ and $q_3 = 3.75\,\text{kg/m}$. The values of the mass moments of inertia of the component links have been determined with the relations: $I_{C_i} = m_i \cdot l_i^2 / 12,\ i = \overline{1,3}$, where the masses $m_i, i = \overline{1,3}$, are given by: $m_i = q_i \cdot l_i, i = \overline{1,3}$, $l_i, i = \overline{1,3}$, being the lengths of the component links. In the case of the leader crank to the value $I_{C_1} = m_1 \cdot l_1^2 / 12$ was added the value of the mass moment of inertia of the flywheel, that in this case is: $I_v = 1\,\text{kg} \cdot \text{m}^2$;
- the technological moment $M_{ru}$ has the following expression:

$$M_{ru} = M_r \cdot \sin\frac{\varphi_1 - \varphi_{1d}}{2} \cdot \sin\frac{\varphi_1 - \varphi_{1a}}{2} \qquad (18)$$

  where: $M_r = 250\,\text{N} \cdot \text{m}$; $\varphi_{1d}$ and $\varphi_{1a}$ are the values of the crank angle $\varphi_1$ corresponding to the two extreme positions of the rocker *3*, when the link *2* is in the prolongation of the crank and when overlap the crank, respectively. In the considered case $\varphi_{1d} = 44.415^\circ$ and $\varphi_{1a} = 218.942^\circ$;
- the variation of the motor moment $M_m$ is given by: $M_m = 301 - 0.2 \cdot \omega_1$ for $\omega_1 \le 5\,\text{rad/s}$ and $M_m = 400 - 20 \cdot \omega_1$ for $\omega_1 > 5\,\text{rad/s}$.

The variation functions depending on $\varphi_1$ corresponding to the angles $\varphi_2$ and $\varphi_3$ (fig. 2) have been determined by solving the following system of equations obtained by projecting on *x* and *y* axes the independent contour: $O - A - B - C - O$ [4]:

$$\begin{cases} l_1 \cdot \cos\varphi_1 + l_2 \cdot \cos\varphi_2 + l_3 \cdot \cos\varphi_3 - l_0 = 0 \\ l_1 \cdot \sin\varphi_1 + l_2 \cdot \sin\varphi_2 + l_3 \cdot \sin\varphi_3 = 0 \end{cases} \tag{19}$$

where: $l_1 = OA$; $l_2 = AB$; $l_3 = BC$; $l_0 = OC$.

Then, the coordinates of the mass centers $x_{C_j}, y_{C_j}, j = \overline{1,3}$, can be calculated with the following relations:

$$\begin{cases} x_{C_1} = OC_1 \cdot \cos\varphi_1; \quad y_{C_1} = OC_1 \cdot \sin\varphi_1 \\ x_{C_2} = l_1 \cdot \cos\varphi_1 + AC_2 \cdot \cos\varphi_2; \quad y_{C_2} = l_1 \cdot \sin\varphi_1 + AC_2 \cdot \sin\varphi_2 \\ x_{C_3} = l_1 \cdot \cos\varphi_1 + l_2 \cdot \cos\varphi_2 + BC_3 \cdot \cos\varphi_3 \\ y_{C_3} = l_1 \cdot \sin\varphi_1 + l_2 \cdot \sin\varphi_2 + BC_3 \cdot \sin\varphi_3 \end{cases} \tag{20}$$

The analytical expressions of the derivatives in relation (11) have been calculated by deriving with respect to $\varphi_1$ the variation functions corresponding to the coordinates of the mass centers $x_{C_j}, y_{C_j}, j = \overline{1,3}$, and to the angles $\varphi_2$ and $\varphi_3$, respectively, using the derivation function *diff* in Maple program [7].

The recursive calculus in relation (17) was started with the value $\omega_1 = 0.1\,\text{rad/s}$.

In fig. 3 the variation of the angular speed $\omega_1$ for five cinematic cycles is presented. In fig. 4 the variation of the angular speed $\omega_1$ in the fourth cinematic cycle, when this variation is already stabilized, is presented.

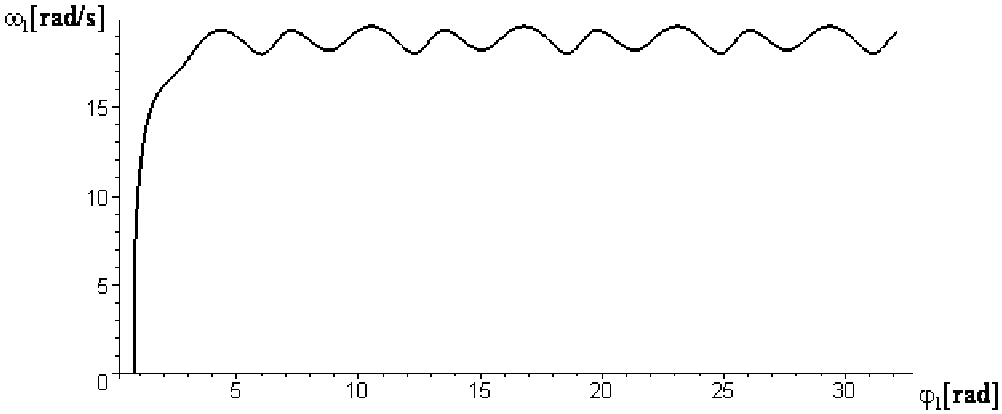


**Fig. 3.** The variation of the angular speed $\omega_1$ for five cinematic cycles

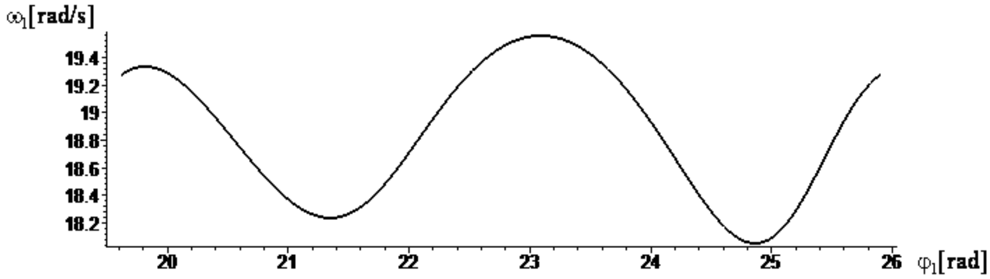


**Fig. 4.** The variation of the angular speed $\omega_1$ in the fourth cinematic cycle

## Conclusions

In this paper a method for establishing and solving the movement equation in the case of the plane mechanisms with one degree of mobility, when the leader element is a crank, have been presented. The method presented is based on the expressing of the dynamic equilibrium in instantaneous powers of all the forces and moments that work on the component links of the mechanism. The variation of the angular speed of the driving crank has been determined by numerically integration of the movement equation using the finite differences method. The results obtained with the simulation program developed in the case of a quadrilateral mechanism highlight the stability of the method used for the integration of the movement equation.

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# Asupra stabilirii şi rezolvării ecuaţiei de mişcare în cazul mecanismelor plane

## Rezumat

*In articol se prezintă o metodă de stabilire şi rezolvare a ecuaţiei de mişcare în cazul mecanismelor plane. Metoda se bazează pe exprimarea echilibrului dinamic în puteri instantanee a tuturor forţelor şi momentelor care lucrează pe elementele componente ale mecanismului. Variaţia vitezei unghiulare a manivelei conducătoare se determină prin integrare numerică a ecuaţiei de mişcare folosind metoda diferenţelor finite. In final, sunt prezentate o serie de rezultate ale simulărilor în cazul unui mecanism patrulater.*